哈密顿力学

§5.5.1 勒让德变换

$F=F(x_1\cdots x_n)$ $dF=\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}dx_i$，令 $u_i=\frac{\partial F}{\partial x_i}$

$\Rightarrow dF=\sum_i u_i dx_i$，若 $J=\left|\frac{\partial(u_1\cdots u_n)}{\partial(x_1\cdots x_n)}\right|\neq 0$，$u_1\cdots u_n$ 也是独立变量

构造 $G=G(u_1,\cdots u_n)=\sum_i x_i u_i-F \Rightarrow dG=\sum_{i=1}^{n}\frac{\partial G}{\partial u_i}du_i$

又 $dG=\sum_i x_i du_i+\sum_i u_i dx_i-dF=\sum_i x_i du_i$

$\Rightarrow x_i=\frac{\partial G}{\partial u_i}$

综上，$\begin{cases} u_i=\frac{\partial F}{\partial x_i} & F=F(x_1,\cdots x_n) \quad G=G(u_1,\cdots u_n) \\ x_i=\frac{\partial G}{\partial u_i} & F+G=\sum_i x_i u_i \end{cases}$ 勒让德变换(基本形式)

√推广：$F=F(x_1,\cdots x_n;y_1,\cdots,y_m)$ $u_i=\frac{\partial F}{\partial x_i}$ （$y_1,\cdots,y_m$ 未参与变换）

构造 $G=G(u_1,\cdots u_n;y_1,\cdots,y_m)=\sum_i x_i u_i-F$

有 $\begin{cases} u_i=\frac{\partial F}{\partial x_i} \\ x_i=\frac{\partial G}{\partial u_i} \end{cases}$ $\frac{\partial F}{\partial y_j}=-\frac{\partial G}{\partial y_j}$

§5.5.2 正则方程

拉格朗日方程 $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\alpha}\right)=\frac{\partial L}{\partial q_\alpha}$ （二阶微分方程）

$L=L(q_1,q_2,\cdots,q_s;\dot q_1,\dot q_2,\cdots,\dot q_s;t)$ 广义动量 $p_\alpha=\frac{\partial L}{\partial \dot q_\alpha}$；$\dot q_\alpha=\frac{\partial H}{\partial p_\alpha}$

$H=H(q_1,q_2\cdots q_s;p_1,p_2\cdots p_s;t)=\sum_\alpha p_\alpha \dot q_\alpha-L$

拉氏方程 $\Rightarrow \frac{d}{dt}p_\alpha=\frac{\partial L}{\partial q_\alpha}=-\frac{\partial H}{\partial q_\alpha}\Rightarrow \dot p_\alpha=-\frac{\partial H}{\partial q_\alpha}$

$\Rightarrow$ 哈密顿正则方程 $\begin{cases} \dot q_\alpha=\frac{\partial H}{\partial p_\alpha} \\ \dot p_\alpha=-\frac{\partial H}{\partial q_\alpha} \end{cases}$ $\alpha=1,2,\cdots,s$ （一阶微分方程组）

eg. 有心力（吸力）取平面极坐标系参数 r,θ 为广义坐标

$L=T-V=\frac{m}{2}(\dot r^2+r^2\dot\theta^2)-V(r)$ $p_r=\frac{\partial L}{\partial \dot r}=m\dot r$，$p_\theta=\frac{\partial L}{\partial\dot\theta}=mr^2\dot\theta$

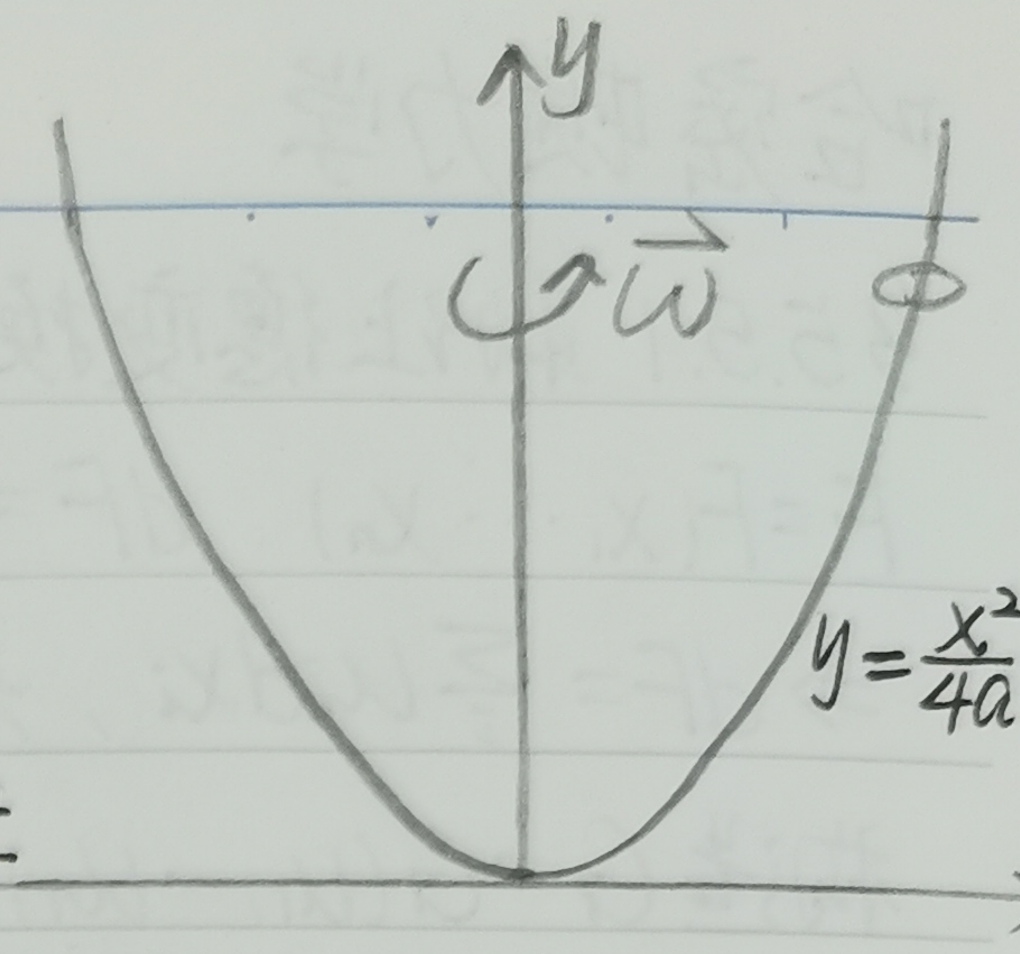
$H=\dot r p_r+\dot\theta p_\theta-L=\frac{m}{2}(\dot r^2+r^2\dot\theta^2)+V(r)=\frac{1}{2}\left(\frac{p_r^2}{m}+\frac{p_\theta^2}{mr^2}\right)+V(r)$

$\begin{cases} \dot r=\frac{\partial H}{\partial p_r}=\frac{p_r}{m}\Rightarrow p_r=m\dot r \\ \dot p_r=-\frac{\partial H}{\partial r}=\frac{p_\theta^2}{mr^3}-\frac{dV}{dr} \end{cases}\Rightarrow m(\ddot r-r\dot\theta^2)=-\frac{dV}{dr}$

$\begin{cases} \dot\theta=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{mr^2}\Rightarrow p_\theta=mr^2\dot\theta \\ \dot p_\theta=-\frac{\partial H}{\partial\theta}=0\Rightarrow p_\theta=h \end{cases}\Rightarrow r^2\dot\theta=h$

牛力√

例题 $q=x,\ L=\frac{m}{2}[(1+\frac{x^2}{4a^2})\dot{x}^2+\omega^2x^2]-mg\frac{x^2}{4a}$



$P_x=\frac{\partial L}{\partial \dot x}=m(1+\frac{x^2}{4a^2})\dot x$

$H=\sum_\alpha P_\alpha \dot q_\alpha - L = \dot x P_x - L$ $\quad H(x,P_x,t)$

$H=\dot x P_x - \frac{m}{2}[(1+\frac{x^2}{4a^2})\dot x^2+\omega^2x^2]+mg\frac{x^2}{4a}$

$=\frac{P_x^2}{m(1+x^2/4a^2)}-\frac{m}{2}(1+\frac{x^2}{4a})\frac{P_x^2}{m^2(1+x^2/4a^2)^2}-\frac{m}{2}\omega^2x^2+mg\frac{x^2}{4a}$

$\Rightarrow H=\frac{1}{2}\frac{P_x^2}{m(1+x^2/4a^2)}-\frac{m}{2}\omega^2x^2+mg\frac{x^2}{4a}$

$\dot x=\frac{\partial H}{\partial P_x}=\frac{P_x}{m(1+x^2/4a^2)}$ ① 回到了原点~ $\Rightarrow \dot P_x=\frac{d}{dt}[m(1+\frac{x^2}{4a^2})\dot x]\leftrightarrow\frac{d}{dt}(\frac{\partial L}{\partial\dot x})$

$\dot P_x=-\frac{\partial H}{\partial x}=+\frac{1}{2}\frac{P_x^2}{m(1+x^2/4a^2)^2}\frac{x}{2a^2}+m\omega^2x-mg\frac{x}{2a}$ ② $\Rightarrow \dot P_x=-\frac{\partial H}{\partial x}\leftrightarrow\frac{\partial L}{\partial x}$

哈密顿正则方程联立结果——拉格朗日方程 废了?逆袭启动!

§5.5.4 相空间 $H=H(q_1,\cdots,q_s;P_1,\cdots;P_s;t)$, $P_\alpha=\frac{\partial L}{\partial\dot q_\alpha}$ $\quad L\not\equiv T-V$

张成一2S维空间——相空间 ⇢ 2S个运动积分?

Game: $\forall f=f(q_1,\cdots;q_s;t)\Rightarrow \dot f=\frac{\partial f}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial f}{\partial q_\alpha}\dot q_\alpha$

$\Rightarrow \frac{\partial\dot f}{\partial\dot q_\alpha}=\frac{\partial}{\partial\dot q_\alpha}(\frac{\partial f}{\partial t}+\sum_{\beta=1}^{s}\frac{\partial f}{\partial q_\beta}\dot q_\beta)=0+\frac{\partial f}{\partial q_\alpha}$ ("消导")

$\Rightarrow\frac{d}{dt}(\frac{\partial\dot f}{\partial\dot q_\alpha})=\frac{d}{dt}(\frac{\partial f}{\partial q_\alpha})=\frac{\partial^2 f}{\partial t\partial q_\alpha}+\sum_{\beta=1}^{s}\frac{\partial^2 f}{\partial q_\alpha\partial q_\beta}\dot q_\beta$

又因为 $\frac{\partial\dot f}{\partial q_\alpha}=\frac{\partial}{\partial q_\alpha}(\frac{\partial f}{\partial t}+\sum_{\beta=1}^{s}\frac{\partial f}{\partial q_\beta}\dot q_\beta)=\frac{\partial^2 f}{\partial t\partial q_\alpha}+\sum_{\alpha=1}^{s}\frac{\partial^2 f}{\partial q_\alpha\partial q_\beta}\dot q_\beta$ $\Big\}\Rightarrow \frac{d}{dt}(\frac{\partial\dot f}{\partial\dot q_\alpha})=\frac{\partial\dot f}{\partial q_\alpha}$

结论: 1) L✓, $L'=L+\dot f$. $\forall f=f(q_1,q_2,\cdots,q_s;t)$ 亦✓

2) $q_\alpha\leftrightarrow P_\alpha=\frac{\partial L}{\partial\dot q_\alpha}\longrightarrow P_\alpha'=\frac{\partial L'}{\partial\dot q_\alpha}=\frac{\partial}{\partial\dot q_\alpha}(L+\dot f)=P_\alpha+\frac{\partial f}{\partial q_\alpha}$ $(-\frac{\partial f}{\partial q_\alpha})$

所以相空间~✓. 正则坐标 $\xleftrightarrow{共轭}$ 正则动量

* 刘维尔定理: 代表点在相空间中运动时, 密度ρ不变.

§5.6.1 泊松括号的定义

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\sum_{\alpha=1}^{s}\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}-\frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right)=:[\varphi,\psi]\ \text{泊松括号}$$

$[\varphi,\psi]=\sum_{\alpha=1}^{s}\left(\frac{\partial\varphi}{\partial q_\alpha}\frac{\partial\psi}{\partial p_\alpha}-\frac{\partial\varphi}{\partial p_\alpha}\frac{\partial\psi}{\partial q_\alpha}\right)$

$\Rightarrow \frac{df}{dt}=\frac{\partial f}{\partial t}+[f,H]$

f是运动积分 $\Longleftrightarrow \frac{\partial f}{\partial t}+[f,H]=0$ eg. $\frac{dH}{dt}=\frac{\partial H}{\partial t}+[H,H]=\frac{\partial H}{\partial t}$

§5.6.2 泊松括号の性质

0) $[\varphi,\varphi]=0$

1) $[C,\psi]=0$，C为常量

2) $[\varphi,\psi]=-[\psi,\varphi]$ 反对称性

3) $[C\varphi,\psi]=C[\varphi,\psi]=[\varphi,C\psi]$

4) $[\varphi,\psi_1+\psi_2]=[\varphi,\psi_1]+[\varphi,\psi_2]$ 分配律

5) $[\varphi,\psi_1\psi_2]=[\varphi,\psi_1]\psi_2+\psi_1[\varphi,\psi_2]$ —结合律

6) $\frac{\partial}{\partial t}[\varphi,\psi]=[\frac{\partial\varphi}{\partial t},\psi]+[\varphi,\frac{\partial\psi}{\partial t}]$ —微商法则

7) $[q_\alpha,\psi]=\frac{\partial\psi}{\partial p_\alpha}$ $[p_\alpha,\psi]=-\frac{\partial\psi}{\partial q_\alpha}$

特别：Hamilton正则方程 $\begin{cases}\dot q_\alpha=\frac{\partial H}{\partial p_\alpha}=[q_\alpha,H]\\ \dot p_\alpha=-\frac{\partial H}{\partial q_\alpha}=[p_\alpha,H]\end{cases}$

8) $[q_\alpha,q_\beta]=0$，$[p_\alpha,p_\beta]=0$，$[q_\alpha,p_\beta]=\delta_{\alpha\beta}$ —基本泊松括号

9) $[f,[\varphi,\psi]]+[\varphi,[\psi,f]]+[\psi,[f,\varphi]]=0$ — 雅克比恒等式

§5.6.3 泊松定理：若力学量f,g均是运动积分，则[f,g]也是运动积分

证：已知 $\frac{df}{dt}=\frac{\partial f}{\partial t}+[f,H]=0$；$\frac{dg}{dt}=\frac{\partial g}{\partial t}+[g,H]=0$

由于 $[H,[f,g]]+[f,[g,H]]+[g,[H,f]]=0$

$\Rightarrow [H,[f,g]]+[f,-\frac{\partial g}{\partial t}]+[g,\frac{\partial f}{\partial t}]=0$

$\Rightarrow [[f,g],H]+[f,\frac{\partial g}{\partial t}]+[\frac{\partial f}{\partial t},g]=0$

$\Rightarrow [[f,g],H]+\frac{\partial}{\partial t}[f,g]=0 \Rightarrow \frac{d}{dt}[f,g]=0$. 证毕.

§5.6.4 例题：J是单个质点の角动量，求$[J_x,J_y]$

解：$[J_x,J_y]=[yp_z-zp_y,zp_x-xp_z]=[yp_z,zp_x]-[yp_z,xp_z]-[zp_y,zp_x]+[zp_y,xp_z]$

其中，$[yp_z,zp_x]=y[p_z,zp_x]+[y,zp_x]p_z=yz[p_z,p_x]+y[p_z,z]p_x+z[y,p_x]p_z+[y,z]p_xp_z$

（$[p_z,p_x]=0$，$[p_z,z]=-1$，$[y,p_x]=0$，$[y,z]=0$）

$=-yp_x$

同理，中二项为0，$[zp_y,xp_z]=xp_y$

$\Rightarrow [J_x,J_y]=xp_y-yp_x=J_z$ 循环～(不出新) 互相推证ing

No.

Date

§5.7 哈密顿原理

§5.7.2 速降线问题 (伽利略→惠更斯→约翰·伯努利)

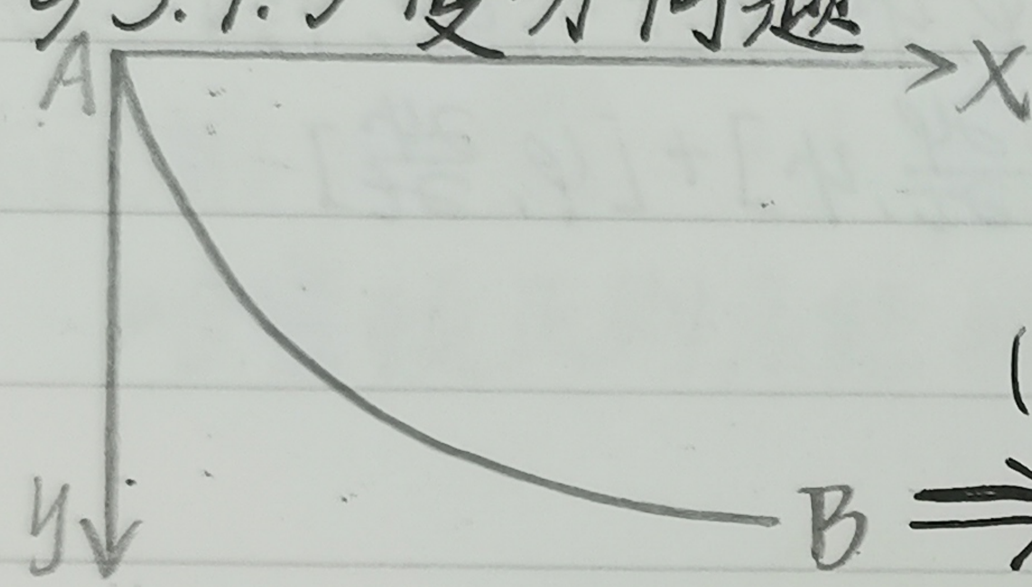
类比: $\frac{\sin\theta_1}{v_1}=\frac{\sin\theta_2}{v_2}=C$ vs $\frac{\sin\theta}{v}=C \Rightarrow \frac{\sin\theta}{\sqrt{2gy}}=C$, 即 $y=C\sin^2\theta=\frac{C}{2}(1-\cos2\theta)$

(光の折射~费马原理v)　又 $\frac{dx}{dy}=\tan\theta, \Rightarrow x=\frac{C}{2}(2\theta-\sin2\theta)$

↓一般化

圆滚线(速降线): $\begin{cases} x=\frac{C}{2}(2\theta-\sin2\theta) \\ y=\frac{C}{2}(1-\cos2\theta) \end{cases}$　联: 古建筑排水~

§5.7.3 变分问题



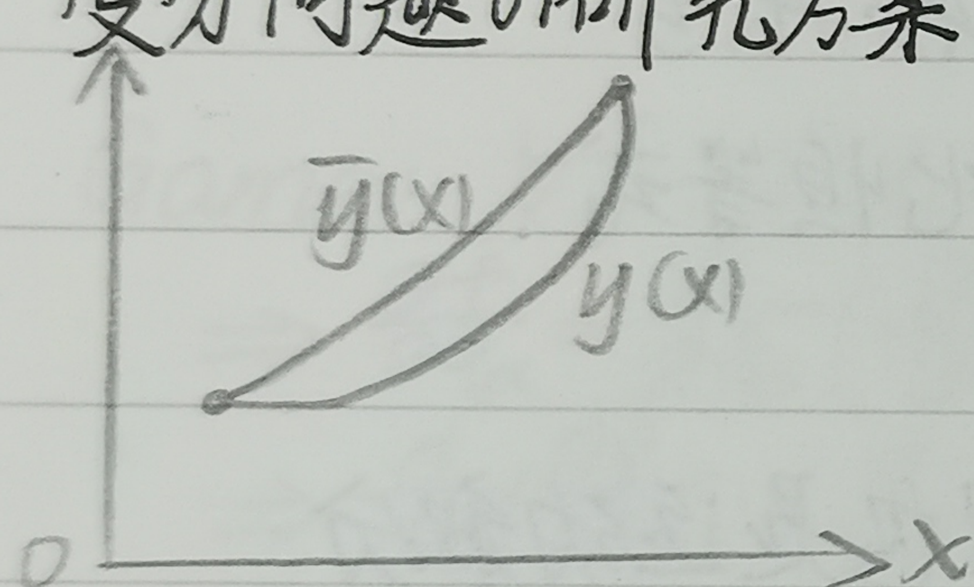
$dt=\frac{ds}{v}=\frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}$, $t=\int_{x_1}^{x_2}\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$　$y(x)=?$ 一困难

(因变量)　(自变量)

$\Rightarrow J=\int_{x_1}^{x_2}F(x,y,y')dx$, $y=y(x)$. $y'=\frac{dy}{dx}$

泛函数: $J=J[y(x)]$ の极值问题——变分问题

变分问题の研究方案



$J=J[y(x)]$. $\bar{J}=\bar{J}[\bar{y}(x)]$　微扰几乎不变?

若 $\delta J=\bar{J}-J=0$, 则 $y(x)$ 为变分问题の解.

$\delta J=\int_{x_1}^{x_2}F(x,\bar{y},\bar{y}')dx-\int_{x_1}^{x_2}F(x,y,y')dx=0$ 极值

§5.7.4 欧拉-拉格朗日方程.

极值条件: $y(x)$ 满足 $\delta J=\bar{J}-\bar{J}=\int_{x_1}^{x_2}F(x,\bar{y},\bar{y}')dx-\int_{x_1}^{x_2}F(x,y,y')dx=0$

$\bar{J}[\bar{y}(x)]=\int_{x_1}^{x_2}F(x,\bar{y},\bar{y}')dx$　注: $\delta x=0, \delta y(x_1)=\delta y(x_2)=0$

$\overset{展开}{=}\int_{x_1}^{x_2}\left(F(x,y,y')+\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)dx$

$\delta J=\bar{J}-J=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)dx=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)\right)dx$

$=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\delta y+\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\delta y\right)-\delta y\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)dx$

$=\frac{\partial F}{\partial y'}\delta y\Big|_{x_1}^{x_2}+\int_{x_1}^{x_2}\delta y\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)dx$

$\delta J=0 \Leftrightarrow \frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$ —— 欧拉-拉格朗日方程.

§5.7.5 哈密顿原理

$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$ 类比 $\frac{\partial L}{\partial q_\alpha}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\alpha}\right)=0$

F, y, x $\qquad$ L, q_α, t

若F不显含x, $\frac{\partial F}{\partial y'}y'-F=C$ $\qquad$ 若L不显含t, $\sum\frac{\partial L}{\partial \dot q_\alpha}\dot q_\alpha-L=C$

· 速降线问题

$F=\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \Rightarrow \frac{1}{\sqrt{2gy}}\frac{y'}{\sqrt{1+y'^2}}-\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}=C \Rightarrow \frac{1}{\sqrt{2gy}}\frac{-1}{\sqrt{1+y'^2}}=C$ ①

由 $\frac{dx}{dy}=\tan\theta$, $\Rightarrow \frac{1}{\sqrt{y}}\sin\theta=C$ ② $\qquad$ ①⇔②, ……

$J=\int_{x_1}^{x_2}F(x,y,y')dx$, $\delta J=0$ 类比 $S=\int_{t_1}^{t_2}L(t,q_\alpha,\dot q_\alpha)dt$, $\delta S=0$

"作用量" —哈密顿原理.

eg. 自由落体运动 $AB=h=\frac{1}{2}gt_0^2$

$L=T-V=\frac{1}{2}m\dot x^2+mgx$, $S=\int_{t_1}^{t_2}Ldt=\int_0^{t_0}(\frac{1}{2}m\dot x^2+mgx)dt$

Reality: $x=\frac{1}{2}gt^2$, $\dot x=gt$, $S_1=\int_0^{t_0}mg^2t^2dt=\frac{1}{3}mg^2t_0^3$ 最小(极小值)

§5.7.6 哈密顿原理の理解.

§5.8.1 正则变换及其条件

· 位形空间の点变换 $L=L(q_1,\cdots,q_s;\dot q_1,\cdots,\dot q_s;t)$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0$.

作变换: $Q_\alpha=Q_\alpha(q_1,\cdots,q_s;t)$ $L'=L'(Q_1,\cdots,Q_s;\dot Q_1,\cdots,\dot Q_s;t)$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot Q_\alpha}\right)-\frac{\partial L}{\partial Q_\alpha}=0$?

eg. 平方反比有心力. $L=\frac{m}{2}(\dot x^2+\dot y^2)-k\frac{1}{\sqrt{x^2+y^2}}$

$r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$. $L'=\frac{m}{2}(\dot r^2+r^2\dot\theta^2)-k\frac{1}{r}$ —可以得到运动积分(≤S个)

· Review: Hamilton正则方程: $\begin{cases}\dot q_\alpha=\frac{\partial H}{\partial p_\alpha}\\ \dot p_\alpha=-\frac{\partial H}{\partial q_\alpha}\end{cases}$ $\alpha=1,2\cdots s$; 相空间: p_α和q_α所张成の空间

正则变换 $\begin{cases}Q_\alpha=Q_\alpha(q_1,\cdots,q_s;p_1,\cdots,p_s;t)\\ P_\alpha=P_\alpha(q_1,\cdots,q_s;p_1,\cdots,p_s;t)\end{cases}$

若∃函数: $K=K(Q_1,\cdots,Q_s;P_1,\cdots,P_s;t)$

s.t. $\begin{cases}\dot Q_\alpha=\frac{\partial K}{\partial P_\alpha}\\ \dot P_\alpha=-\frac{\partial K}{\partial Q_\alpha}\end{cases}$ 可以反映运动规律, 则(正确的)

哈密顿原理: $\delta\int_{t_1}^{t_2}Ldt=0$.

$\delta\int_{t_1}^{t_2}(\sum_\alpha p_\alpha\dot q_\alpha-H)dt=0 \Leftrightarrow \begin{cases}\dot q_\alpha=\frac{\partial H}{\partial p_\alpha}\\ \dot p_\alpha=-\frac{\partial H}{\partial q_\alpha}\end{cases}$

$\delta\int_{t_1}^{t_2}(\sum_\alpha P_\alpha\dot Q_\alpha-K)dt=0 \Leftrightarrow \begin{cases}\dot Q_\alpha=\frac{\partial K}{\partial P_\alpha}\\ \dot P_\alpha=-\frac{\partial K}{\partial Q_\alpha}\end{cases}$

$\Rightarrow \sum_\alpha p_\alpha\dot q_\alpha-H=\sum_\alpha P_\alpha\dot Q_\alpha-K+\frac{dU}{dt}$

U: 正则变换の生成(母)函数

§5.8.2 正则变换×4.

正则变换の条件：$dU=\sum_\alpha p_\alpha dq_\alpha-\sum_\alpha P_\alpha dQ_\alpha+(K-H)dt$ ①

$U_1=U_1(q,Q,t)$ $U_2=U_2(q,P,t)$ $U_3=U_3(p,Q,t)$. $U_4=U_4(p,P,t)$

1) $U_1=U_1(q_1,\cdots,q_s;Q_1,\cdots,Q_s;t)$ 令 $U_1=U$

$dU_1=\sum_\alpha\frac{\partial U_1}{\partial q_\alpha}dq_\alpha+\sum_\alpha\frac{\partial U_1}{\partial Q_\alpha}dQ_\alpha+\frac{\partial U_1}{\partial t}dt$ ② $\overset{①②}{\Longrightarrow}\begin{cases}p_\alpha=\frac{\partial U_1}{\partial q_\alpha}\\P_\alpha=-\frac{\partial U_1}{\partial Q_\alpha}\end{cases}$ $(p,q\sim P,Q)$ $\alpha=1,2,\cdots s.$

$K=H+\frac{\partial U_1}{\partial t}=K(Q,P,t)$ $\begin{cases}\dot{Q}_\alpha=\frac{\partial K}{\partial P_\alpha}\\\dot{P}_\alpha=-\frac{\partial K}{\partial Q_\alpha}\end{cases}\Rightarrow\begin{cases}Q(t)=\\P(t)=\end{cases}$ ✓

2) $U_2=U_2(q_1,\cdots,q_s;P_1,\cdots,P_s;t)$ 令 $U_2=U+\sum_\alpha P_\alpha Q_\alpha$ ③

$dU_2=\sum_\alpha\frac{\partial U_2}{\partial q_\alpha}dq_\alpha+\sum_\alpha\frac{\partial U_2}{\partial P_\alpha}dP_\alpha+\frac{\partial U_2}{\partial t}dt$ ④

$\overset{①③}{\Longrightarrow}=\sum_\alpha p_\alpha dq_\alpha-\sum_\alpha P_\alpha dQ_\alpha+(K-H)dt+\sum_\alpha P_\alpha dQ_\alpha+\sum_\alpha Q_\alpha dP_\alpha$ ⑤

$\overset{④⑤}{\Longrightarrow}\begin{cases}p_\alpha=\frac{\partial U_2}{\partial q_\alpha}\\Q_\alpha=\frac{\partial U_2}{\partial P_\alpha}\end{cases}$ $\alpha=1,2\ldots s$ $K=H+\frac{\partial U_2}{\partial t}$ | 同理 $\begin{cases}q_\alpha=-\frac{\partial U_3}{\partial p_\alpha}\\P_\alpha=-\frac{\partial U_3}{\partial Q_\alpha}\end{cases}$ $\alpha=1,2,\cdots s$ $K=H+\frac{\partial U_3}{\partial t}$

§5.8.3 正则变换例题

例1. $U=\sum_\beta p_\beta P_\beta$ 解：$\begin{cases}q_\alpha=-\frac{\partial U}{\partial p_\alpha}=-P_\alpha\\Q_\alpha=\frac{\partial U}{\partial P_\alpha}=p_\alpha\end{cases}$ $\leftarrow K=H$ $\Rightarrow\begin{cases}Q_\alpha=p_\alpha\\P_\alpha=-q_\alpha\end{cases}$ 可以么？

检验：$\begin{cases}\dot{Q}_\alpha=\frac{\partial K}{\partial P_\alpha}\Leftrightarrow\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}\\\dot{P}_\alpha=-\frac{\partial K}{\partial Q_\alpha}\Leftrightarrow\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha}\end{cases}$ ✓ (Hamilton正则方程

例2. 一维谐振子：$H=\frac{p^2}{2m}+\frac{m}{2}\omega^2q^2$ $U=\frac{m}{2}\omega q^2\,\mathrm{ctg}Q$ (cot) ⟶ 如何设计？TBC…

解：$\begin{cases}p=\frac{\partial U}{\partial q}=m\omega q\cot Q & ①\\P=-\frac{\partial U}{\partial Q}=+\frac{m}{2}\omega q^2\csc^2\theta & ②\end{cases}$

$K=H=\frac{1}{2m}m^2\omega^2q^2\cot^2\theta+\frac{1}{2}m\omega^2q^2=\frac{m}{2}\omega^2q^2\csc^2Q=\omega P$

$\begin{cases}\dot{Q}=\frac{\partial K}{\partial P}=\omega\Rightarrow Q=\omega t+\varphi\\\dot{P}=-\frac{\partial K}{\partial Q}=0\Rightarrow P=C\end{cases}$ 由②得 $q^2=\frac{2P}{m\omega}\sin^2Q$,

$q=A\sin Q=A\sin(\omega t+\varphi)$

哈密顿-雅克比方程

· 正则变换の目的 $\begin{cases}\dot{Q}_\alpha = \frac{\partial K}{\partial P_\alpha} \\ \dot{P}_\alpha = \frac{-\partial K}{\partial Q_\alpha}\end{cases}$ $K = K(Q_1, \cdots, Q_s; P_1, \cdots, P_s; t)$

目标：使K尽量少含 Q_α, P_α.

Best: $K = 0 \Rightarrow \begin{cases}\dot{Q}_\alpha = 0 \\ \dot{P}_\alpha = 0\end{cases} \Rightarrow \begin{cases}Q_\alpha = A_\alpha \text{(常数)} \\ P_\alpha = B_\alpha\end{cases} \Rightarrow q_\alpha = q_\alpha(Q_1, \cdots, Q_s; P_1, \cdots, P_s; t)$

问：能做到吗？怎么做呢？

(答) $\left.\begin{array}{l} K = 0 \\ K = H + \frac{\partial U}{\partial t}\end{array}\right\} \Rightarrow H(q_1 \cdots q_s; P_1 \cdots P_s; t) + \frac{\partial U}{\partial t} = 0$

例：$U = U(q_1 \cdots q_s; P_1 \cdots P_s; t)$ $\begin{cases}P_\alpha = \frac{\partial U}{\partial q_\alpha} \\ Q_\alpha = \frac{\partial U}{\partial P_\alpha}\end{cases}$ （II类正则变换）

$H(q_1, \cdots, q_s; \frac{\partial U}{\partial q_1}, \cdots, \frac{\partial U}{\partial q_s}; t) + \frac{\partial U}{\partial t} = 0$ —— 哈-雅方程.

方程の解U包含 s+1 个积分常数：$U = C_{s+1} + f(q_1, \cdots, q_s; C_1, \cdots, C_s; t)$

*例：一维谐振子：$H = \frac{1}{2m}P^2 + \frac{1}{2}kx^2$

解：$U = U(x, P)$，$H(x, \frac{\partial U}{\partial x}) = -\frac{\partial U}{\partial t}$

$\Rightarrow U(x, P) = -Pt + \frac{1}{2}x\sqrt{2mP - mkx^2} + P\sqrt{\frac{m}{k}}\sin^{-1}(\sqrt{\frac{k}{2P}}x) + A$

$\begin{cases}p = \frac{\partial U}{\partial x} = \sqrt{2mP - mkx^2} \\ Q = \frac{\partial U}{\partial P} = -t + \sqrt{\frac{m}{k}}\sin^{-1}(\sqrt{\frac{k}{2m}}x)\end{cases}$ $K = 0 \Rightarrow \begin{cases}P = C_2 \\ Q = C_1\end{cases}$

$\Rightarrow x = \sqrt{\frac{2P}{k}}\sin[\sqrt{\frac{k}{m}}(Q+t)] = \sqrt{\frac{2P}{k}}\sin[\sqrt{\frac{k}{m}}t + \sqrt{\frac{k}{m}}Q]$. P.Q由初始条件确定~

· 哈-雅方程解(U)の意义

$H(q_1, \cdots, q_s; \frac{\partial U}{\partial q_1}, \cdots, \frac{\partial U}{\partial q_s}; t) + \frac{\partial U}{\partial t} = 0$

$U = U(q_1, \cdots, q_s; P_1, \cdots, P_s; t) = C_{s+1} + f(q_1, \cdots, q_s; C_1, \cdots, C_s; t)$

$\Rightarrow \frac{dU}{dt} = \frac{\partial U}{\partial t} + \sum_{\alpha=1}^{s}\frac{\partial U}{\partial q_\alpha}\dot{q}_\alpha = -H + \sum_{\alpha=1}^{s}P_\alpha\dot{q}_\alpha = L$ (拉~)

$U = S \rightarrow$ 系统の作用量！

完结撒花！