南京大学
地址：南京市仙林大道163号　邮编：210046
网址：http://www.nju.edu.cn

统计物理

一、系综

$S=-k\sum_s \rho_s \ln \rho_s$

A_1 | A_2　$A^{(0)}$

$\Omega^{(0)}(E_1,E_2)=\Omega_1(E_1)\Omega_2(E_2)$

$E_1+E_2=E^{(0)}$

微正则

$\frac{\partial \Omega^{(0)}}{\partial E_1}=0 \Rightarrow \frac{\partial \ln \Omega_1(E_1)}{\partial E_1}=\frac{\partial \ln \Omega_2(E_2)}{\partial E_2}$

$\beta=\left(\frac{\partial \ln \Omega(N,E,V)}{\partial E}\right)_{N,V}$　　$\beta_1=\beta_2$

热力学中 $\left(\frac{\partial S_1}{\partial U_1}\right)_{N_1,V_1}=\left(\frac{\partial S_2}{\partial U_2}\right)_{N_2,V_2} \Rightarrow \frac{1}{T_1}=\frac{1}{T_2}$

$\sim \begin{cases}\beta=\frac{1}{kT}\\ S=k\ln\Omega\end{cases}$　　$\rho_s=\frac{1}{\Omega}$

2. E_s 系统 | 热源 $E^{(0)}-E_s$

$\rho_s \propto \Omega_r(E^{(0)}-E_s)$

正则

展开 $\ln\Omega_r(E^{(0)}-E_s)=\ln\Omega_r(E^{(0)})+\left(\frac{\partial \ln \Omega_r}{\partial E_r}\right)_{E_r=E^{(0)}}(-E_s)$

$=\ln\Omega_r(E^{(0)})-\beta E_s$

$\rho_s \propto e^{-\beta E_s}$　归一化系数 $Z=\sum_s e^{-\beta E_s}$

$\begin{cases} Z=\sum_s e^{-\beta E_s}=\sum_l \Omega_l e^{-\beta E_l} \\ Z=\frac{1}{N!\,h^{Nr}}\int e^{-\beta E(q,p)}\,d\Omega \end{cases}$　　$\rho_s=\frac{1}{Z}e^{-\beta E_s}$

$\rho(q,p)\,d\Omega=\frac{1}{N!\,h^{Nr}}\frac{e^{-\beta E(q,p)}}{Z}\,d\Omega$

① 内能 $\bar{E}$

$U=\frac{1}{Z}\sum_s E_s e^{-\beta E_s}=\frac{1}{Z}\left(-\frac{\partial}{\partial\beta}\right)\left(\sum_s e^{-\beta E_s}\right)=-\frac{\partial}{\partial\beta}\ln Z$

② 广义力 $\overline{\left(\frac{\partial E_s}{\partial y}\right)}$

$Y=\frac{1}{Z}\sum_s \frac{\partial E_s}{\partial y}e^{-\beta E_s}=\frac{1}{Z}\left(-\frac{1}{\beta}\frac{\partial}{\partial y}\right)\left(\sum_s e^{-\beta E_s}\right)=-\frac{1}{\beta}\frac{\partial}{\partial y}\ln Z$

③ 熵

$d\ln Z=\frac{\partial}{\partial\beta}\ln Z\,d\beta+\frac{\partial}{\partial y}\ln Z\,dy$

$\beta(dU-Y\,dy)=-\beta\,d\left(\frac{\partial}{\partial\beta}\ln Z\right)+\frac{\partial}{\partial y}\ln Z\,dy=d\left(\ln Z-\beta\frac{\partial}{\partial\beta}\ln Z\right)$

而 $\frac{1}{T}(dU-Y\,dy)=dS$　　$\begin{cases} S=k\left(\ln Z-\beta\frac{\partial}{\partial\beta}\ln Z\right)\\ \beta=\frac{1}{kT}\end{cases}$　　$F=-kT\ln Z$

3. 系统 | 源　E_r, N_r;　$E^{(0)}-E_r$, $N^{(0)}-N_r$

巨正则

$\rho_{Ns}\propto \Omega_r(N^{(0)}-N,E^{(0)}-E_s)$

展开 $\ln\Omega_r(N^{(0)}-N,E^{(0)}-E_s)$

$=\ln\Omega_r(N^{(0)},E^{(0)})+\left(\frac{\partial \ln\Omega_r}{\partial N_r}\right)_{N_r=N^{(0)}}(-N)$

$+\left(\frac{\partial \ln\Omega_r}{\partial E_r}\right)_{E_r=E^{(0)}}(-E_s)$

$=\ln\Omega_r(N^{(0)},E^{(0)})-\alpha N-\beta E_s$

$\rho_{Ns}\propto e^{-\alpha N-\beta E_s}$

归一化系数 $\Xi=\sum_{N=0}^{\infty}\sum_S e^{-\alpha N-\beta E_s}$

$\Xi=\sum_{N=0}^{\infty}\sum_s e^{-\alpha N-\beta E_s}$　　$\rho_{Ns}=\frac{1}{\Xi}e^{-\alpha N-\beta E_s}$

$\Xi=\sum_N \frac{e^{-\alpha N}}{N!\,h^{Nr}}\int e^{-\beta E(q,p)}\,d\Omega$　　$\rho_N\,dq\,dp=\frac{1}{N!\,h^{Nr}\Xi}e^{-\alpha N}\ldots d\Omega$

① 粒子数 $\bar{N}$

$\bar{N}=-\frac{\partial}{\partial\alpha}\ln\Xi$

② 内能 $\bar{E}$

$U=-\frac{\partial}{\partial\beta}\ln\Xi$

③ 广义力

$Y=-\frac{1}{\beta}\frac{\partial}{\partial y}\ln\Xi$

④ 熵

$S=k\left(\ln\Xi-\alpha\frac{\partial\ln\Xi}{\partial\alpha}-\beta\frac{\partial\ln\Xi}{\partial\beta}\right)$

二、箱归一化

$$\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3) \sim dk_1 dk_2 dk_3 = \left(\frac{2\pi}{L}\right)^3 dn_1 dn_2 dn_3$$

1. 非相对论 $\varepsilon = \frac{p^2}{2m}$

$$p = \frac{2n\pi}{L}\hbar \qquad \varepsilon_{n_x n_y n_z} = \frac{1}{2m}\left(\frac{2\pi\hbar}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

2. 相对论 $\varepsilon = cp$

$$\varepsilon_{n_x n_y n_z} = c \cdot \frac{2\pi\hbar}{L}(n_x^2 + n_y^2 + n_z^2)^{\frac{1}{2}}$$

逻辑 $\dfrac{\text{超球壳内态数}}{\text{相空间体元}}$

3. D.O.S. 态密度 (k空间)

$$D(\varepsilon)d\varepsilon = dN = g\frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} \underset{\varepsilon = \frac{(\hbar k)^2}{2m}}{=\!=} g\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}d\varepsilon$$

$$\underset{\varepsilon = c\hbar k}{=\!=} g \cdot \frac{4\pi V\varepsilon^2 d\varepsilon}{(ch)^3}$$

① 3 dim $\boxed{V = L^3}$ $p_x \to p_x + dp_x$, $p_y \to p_y + dp_y$, $p_z \to p_z + dp_z$ $\Rightarrow \frac{V}{h^3}dp_x dp_y dp_z \to \frac{4\pi V}{h^3}p^2 dp$

② 2 dim $\boxed{A = L^2}$ $p_x \to p_x + dp_x$, $p_y \to p_y + dp_y$ $\quad \frac{A\, dp_x dp_y}{h^2} \to \frac{2\pi A}{h^2}p\, dp$

③ 1 dim $\boxed{L}$ $p \to p + dp$ 两方向 $\quad \frac{2L\, dp}{h}$

三、近独立粒子平均分布

$$\rho_{Ns} = \frac{1}{\Xi}e^{-\alpha N - \beta E_s}$$

$$\Xi = \sum_{N=0}^{\infty}\sum_s e^{-\alpha N - \beta E_s}$$

$$N = \sum_l a_l \qquad E = \sum_l \varepsilon_l a_l$$

$$\begin{cases} \text{Bose} & \Omega_{B\text{-}E} = \prod_l \frac{(\omega_l + a_l - 1)!}{(\omega_l - 1)!\, a_l!} \\ \text{Fermi} & \Omega_{F\text{-}D} = \prod_l \frac{\omega_l!}{a_l!(\omega_l - a_l)!} \end{cases}$$

$$\Xi = \sum_N \sum_s e^{-\alpha N - \beta E_s} = \sum_{\{a_l\}} \Omega e^{-\sum_l (\alpha + \beta\varepsilon_l)a_l}$$

$$= \underset{\text{换序}}{\sum_{\{a_l\}}} \prod_l \Omega_l e^{-(\alpha + \beta\varepsilon_l)a_l} = \prod_l \Xi_l$$

$$\binom{-m}{n} = (-1)^n\binom{m+n-1}{n}$$

$$(1-x)^{-m} = \sum_{n=0}^{\infty}\binom{m+n-1}{n}x^n$$

① B-E

$$\Xi_l = \sum_{a_l=0}^{\infty}\binom{\omega_l + a_l - 1}{a_l}e^{-(\alpha + \beta\varepsilon_l)a_l} = \left[1 - e^{-(\alpha + \beta\varepsilon_l)}\right]^{-\omega_l}$$

$$\bar{a}_l = -\frac{\partial}{\partial\alpha}\ln\Xi_l = \frac{\omega_l}{e^{\alpha + \beta\varepsilon_l} - 1}$$

② F-D

$$\Xi_l = \sum_{a_l=0}^{\omega_l}\binom{\omega_l}{a_l}e^{-(\alpha + \beta\varepsilon_l)a_l} = \left[1 + e^{-(\alpha + \beta\varepsilon_l)}\right]^{\omega_l}$$

$$\bar{a}_l = -\frac{\partial}{\partial\alpha}\ln\Xi_l = \frac{\omega_l}{e^{\alpha + \beta\varepsilon_l} + 1}$$

○非简并条件

$$n\lambda^3 = \frac{N}{V} h^3 \left(\frac{1}{2\pi mkT}\right)^{\frac{3}{2}} \ll 1$$

热波长 $\lambda = \frac{h}{\sqrt{2m\varepsilon}} \sim h\left(\frac{1}{2\pi mkT}\right)^{\frac{1}{2}}$

四、光子气体　光子数不守恒

① $a_l = \frac{\omega_l}{e^{\beta\varepsilon_l}-1}$

$$g\frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} \xrightarrow{p=\hbar k} \frac{8\pi V p^2}{h^3}dp \xrightarrow[\varepsilon=\hbar\omega]{\varepsilon=cp} \frac{V}{\pi^2c^3}\omega^2 d\omega$$

$$U(\omega,T)d\omega = \frac{V}{\pi^2c^3}\frac{\hbar\omega^3}{e^{\frac{\hbar\omega}{kT}}-1}d\omega$$

$$\Downarrow U = \frac{\pi^2k^4}{15c^3\hbar^3}VT^4$$

$\rightarrow$ wein：$\frac{d}{dx}\left(\frac{x^3}{e^x-1}\right)=0$

② 另解 $\ln\Xi = -\sum_l \omega_l \ln(1-e^{-\alpha-\beta\varepsilon_l})$

$$= -\frac{V}{\pi^2c^3}\int_0^\infty \omega^2 \ln(1-e^{-\beta\hbar\omega})d\omega.$$

$$U = -\frac{\partial}{\partial\beta}\ln\Xi$$

$\leadsto$ B-E凝聚

$$a_l = \frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1}$$

$$\frac{1}{V}\sum_l \frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1} = \frac{N}{V} = n$$

$$\Rightarrow \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \frac{\varepsilon^{\frac{1}{2}}d\varepsilon}{e^{\frac{\varepsilon-\mu}{kT}}-1} = n$$

五、电子气体

$$D(\varepsilon)d\varepsilon = \frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}d\varepsilon$$

$$\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \frac{\varepsilon^{\frac{1}{2}}d\varepsilon}{e^{\frac{\varepsilon-\mu}{kT}}+1} = N$$

$$f = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1} \begin{cases} =1, & \varepsilon < \mu(0) \\ =0, & \varepsilon > \mu(0) \end{cases}$$

$$\mu(0) = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}}$$

$$\begin{cases} p_F = \sqrt{2m\mu(0)} \\ kT_F = \mu(0) \\ U(0) = \frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^{\mu(0)} \varepsilon^{\frac{3}{2}}d\varepsilon = \frac{3N}{5}\mu(0) \\ p(0) = \frac{2}{3}\frac{U(0)}{V} = \frac{2}{5}n\mu(0) \end{cases}$$

$p = -\sum_l a_l \frac{\partial\varepsilon_l}{\partial V}$ 视不同情况