2023.12.15 费米子和玻色子

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1. 激光.

① 自发辐射：高能级的电子"自发"到低能级

注：量子电动力学解释为量子涨落. $E_n = n\hbar\omega + \frac{1}{2}\hbar\omega$.

② 受激辐射.

$n \propto e^{-\beta E}$, $\beta = \frac{1}{k_B T}$. 波尔兹曼分布.

在平衡状态下，$n_2 \ll n_1$，$E_2 > E_1$.

若可以满足 $n_2 \gg n_1$，$E_2 > E_1$，此时便可以发射出激光.

2. X射线.

用强光把基态的电子打出，产生一个空穴，外层的电子向内层填充，产生的光即为 X-射线.

$$\lambda = \frac{hc}{E_K}$$

在伦琴实验中，$E_K = 35\,keV$，$\lambda_{min} = 35.4\,pm$.

X射线衍射：$2d\sin\theta = n\lambda$.

第二十七章 费米统计与玻色统计.

1. 全同性

粒子的运动满足：① 量子化、离散化 ② 全同性（假定公设）：同类原子是不可区分的.

宏观物体是可识别的（因为宏观物体有轨道）.

$\psi(\vec{r}_1, \vec{r}_2, t) \xrightarrow{\text{两个粒子交换}} \tilde{\psi}(\vec{r}_2, \vec{r}_1, t)$

$|\psi(\vec{r}_1, \vec{r}_2, t)|^2 = |\tilde{\psi}(\vec{r}_2, \vec{r}_1, t)|^2$，即新状态与旧状态相现率密度完全相同. 全同性.

$\Rightarrow \tilde{\psi}(\vec{r}_2, \vec{r}_1, t) = \pm\psi(\vec{r}_1, \vec{r}_2, t)$. $\begin{cases} \text{玻色子取 } + \\ \text{费米子取 } - \end{cases}$

若 $S = \frac{1}{2}\hbar, \frac{3}{2}\hbar, \frac{5}{2}\hbar, \cdots$ 费米子

$S = 0\hbar, 1\hbar, 2\hbar, \cdots$ 玻色子 （自旋角动量）.

即：具有半奇数自旋的粒子是费米子，具有整数自旋的粒子是玻色子.

多粒子系统 中子气体 光子气体.

2. 量子化.

一维深势阱：$U(x) = \begin{cases} 0, & 0 < x < \varepsilon \\ +\infty, & \text{otherwise} \end{cases}$

二维深势阱：$U(x,y) = \begin{cases} 0, & 0 < x < a,\ 0 < y < b \\ +\infty, & \text{otherwise} \end{cases}$

三维深势阱. $U(x,y,z) = \begin{cases} 0, & 0 < x < a,\ 0 < y < b,\ 0 < z < c \\ +\infty, & \text{otherwise} \end{cases}$

(x) 在二维时，$H = -\frac{\hbar^2}{2m}\nabla^2 + U(x,y) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + U(x,y)$

$0 < x < a$，$0 < y < b$，$U = 0$，$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

otherwise. $U = +\infty$. $|\psi(x,y)|^2 dx\,dy = 0 \Rightarrow |\psi(x,y)|^2 = 0 \Rightarrow \psi(x,y) = 0$.

阱内. $\hat{H}\psi = E\psi$. $\rightarrow$ $-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = E\psi(x,y)$.

演算过程 ① $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\frac{2mE}{\hbar^2}\psi$. 令 $k = \sqrt{\frac{2mE}{\hbar^2}}$，有

$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -k^2\psi$.

分离变量，有 $\psi(x,y) = \psi(x)\psi(y) \longrightarrow \psi(y)\frac{\partial^2\psi(x)}{\partial x^2} + \psi(x)\frac{\partial^2\psi(y)}{\partial y^2} = -k^2\psi(x)\psi(y)$

$\longrightarrow \frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} = -k^2 - \frac{1}{\psi(y)}\frac{\partial^2\psi(y)}{\partial y^2} = -k_1^2$. 令 $k_2^2 = k^2 - k_1^2$，有 $k_1^2 + k_2^2 = k^2$.

解得 $\frac{\partial^2\psi(x)}{\partial x^2} + k_1^2\psi(x) = 0$，$\frac{\partial^2\psi(y)}{\partial y^2} + k_2^2\psi(y) = 0$.

② 边界条件：在阱的四条边上为 0.

(i) $\psi(x,y)|_{x=0} = \psi(x)\psi(y)|_{x=0} = 0$. $\Rightarrow \psi(x)|_{x=0} = 0$.

(ii) $\psi(x)|_{x=a} = 0$ (iii) $\psi(y)|_{y=0} = 0$ (iv) $\psi(y)|_{y=b} = 0$.

总结：① $\frac{\partial^2\psi(x)}{\partial x^2} + k_1^2\psi(x) = 0$，$\psi(x)|_{x=0} = \psi(x)|_{x=a} = 0$.

② $\frac{\partial^2\psi(y)}{\partial y^2} + k_2^2\psi(y) = 0$. $\psi(y)|_{y=0} = \psi(y)|_{y=b} = 0$.

通解：代入初始条件：

$\psi(x) = A\sin\left(\frac{k\pi}{a}x\right)$，$A \neq 0$，$k = 0, 1, 2, \cdots$.

$\psi(y) = A'\sin\left(\frac{k\pi}{b}y\right)$，$A' \neq 0$，$k = 0, 1, 2 \cdots$.

归一化条件 $\left(k_n = \frac{n\pi}{a},\ k_m = \frac{m\pi}{b}\right)$

$\int_0^a |\psi_n(x)|^2 dx = 1 \Rightarrow \psi_n(x) = \sqrt{\frac{2}{a}}\sin(k_n x)$.

$\int_0^b |\psi_m(y)|^2 dy = 1 \Rightarrow \psi_m(y) = \sqrt{\frac{2}{b}}\sin(k_m y)$

$\therefore \psi_{nm}(x,y) = \frac{2}{\sqrt{ab}}\sin(k_n x)\sin(k_m y)$，$E = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}(k_n^2 + k_m^2)$.

三维深势阱类比上述过程即可，这里略之.

三维深势阱解方程：

$\psi_{nml}(x,y,z) = 2\sqrt{\frac{2}{abc}}\sin(k_n x)\sin(k_m y)\sin(k_l z)$.

$\therefore E_{n,m,l} = \frac{\hbar^2}{2m}(k_n^2 + k_m^2 + k_l^2)$.

内能密度 $\frac{U}{V} = \frac{1}{V}\sum_k n_k \cdot \varepsilon_k$（能量，平均粒子能量）

粒子密度 $\frac{N}{V} = \frac{1}{V}\sum_k n_k$.

记 $\vec{k} = k_n\vec{e}_x + k_m\vec{e}_y + k_l\vec{e}_z$.

$\Delta\vec{k}^3 = \Delta k_1\Delta k_2\Delta k_3 = \frac{\pi^3}{abc}$，$\therefore \Delta k = \frac{\pi}{d}$（体积元）.

当 $V \to \infty$，$\Delta\vec{k} \to 0$. 并用 $f(\vec{k})$ 统一表示上述前量

$\therefore \frac{1}{V}\sum_{\vec{k}} f(\vec{k})\Delta\vec{k} = \frac{1}{\pi^3}\sum_{\vec{k}} f(\vec{k})\Delta\vec{k}$（黎曼和）

由于 $\int_a^b f(x)dx = \lim_{|\Delta x_i| \to 0}\sum_i f(\xi_i)\Delta x_i$

$\therefore \lim_{V \to +\infty}\frac{1}{V}\sum_{\vec{k}} f(\vec{k}) = \frac{1}{\pi^3}\int_V f(\vec{k})d\vec{k} = \frac{1}{\pi^3}\int_V f(\vec{k})\,dk_1 dk_2 dk_3$.

$= \frac{1}{\pi^3}\int_0^{+\infty} dk_1 \int_0^{+\infty} dk_2 \int_0^{+\infty} f(\vec{k})\,dk_3$.

又 $f(\vec{k}) = f(|\vec{k}|) = f(\sqrt{k_1^2 + k_2^2 + k_3^2})$ 偶函数.

故 $\lim_{V \to +\infty}\frac{1}{V}\sum_{\vec{k}} f(\vec{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} f(\vec{k})\,dk_3$.

$\boxed{\therefore \frac{1}{V}\sum_{\vec{k}} f(\vec{k}) = \int_V \frac{d\vec{k}}{(2\pi)^3} f(\vec{k})，V\text{为全空间}.}$

这一部分扩展.（xmw 缺点、较难）

若记 $dk_1 dk_2 dk_3 = k^2\sin\theta\,d\theta\,d\varphi$

上述积分化为 $LHS = \frac{4\pi}{(2\pi)^3}\int_{-\infty}^{+\infty} dk \cdot k^2 f(k)$.

由于 $k = \sqrt{\frac{2mE}{\hbar^2}}$，

$\therefore$ 原积分化为 $LHS = \frac{4\pi}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\int_0^{+\infty} d\sqrt{E}\ \ Ef(E)$

$\therefore \frac{1}{V}\sum_{\vec{k}} f(\vec{k}) = \frac{4\pi}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \cdot \frac{1}{2}\int_0^{+\infty}\sqrt{E}\,f(E)\,dE$

$= \frac{2\pi}{h^3}(2m)^{\frac{1}{2}}\int_0^{+\infty}\sqrt{E}\,f(E)\,dE$

$= \int_0^{+\infty} dE\ g(E)f(E)$. 称 $g(E) = \frac{2\pi}{h^3}(2m)^{\frac{3}{2}}\sqrt{E}$ 为态密度

线密度：$\frac{U}{a} = \frac{1}{a}\sum_k n_k\varepsilon_k$，$\frac{N}{a} = \frac{1}{a}\sum_k n_k$，$k = n\frac{\pi}{a}$.

$\therefore$ 故 $\sum_k f(k) = \frac{1}{\pi}\sum_k f(k)\Delta k = \frac{1}{\pi}\int_0^{+\infty} f(k)\,dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(k)\,dk$.

二维：$\frac{1}{S}\sum_{\vec{k}} f(\vec{k}) = \frac{1}{(2\pi)^2}\int_V f(\vec{k})d\vec{k}$. V 为全空间.