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Name _____

January 10, 2020 Friday

SID _____

School/Department _____

University Physics

Final Examination

Kuang Yaming Honors School, Nanjing University

$$\Gamma(\nu) = \int_0^{+\infty} e^{-t} t^{\nu-1} dt, \quad \Gamma(\nu+1) = \nu\Gamma(\nu), \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}; \quad \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

c	h	ħ	e	m _e	m _n	u	N _A	k _B
3.0×10 ⁸ m/s	6.63×10 ⁻³⁴ J·s 1.24×10 ³ eV·nm/c	1.05×10 ⁻³⁴ J·s	1.6×10 ⁻¹⁹ C	9.11×10 ⁻³¹ kg 0.511MeV/c ²	939.565 MeV/c ²	931.494 MeV/c ²	6.02×10 ²³ mol ⁻¹	8.617×10 ⁻⁵ eV/K

Quantum numbers of three quarks:

Quark symbol	Charge number	Spin	Baryon number	Strangeness
u	2/3	1/2	1/3	0
d	-1/3	1/2	1/3	0
s	-1/3	1/2	1/3	-1

Select five out of the following six problems.

1.(20 pts.) If the electron of a hydrogen atom is replaced by a tau (τ) lepton, a hydrogen tau atom is formed. The rest mass of tau lepton is $m_\tau = 1777.05 \text{ MeV}/c^2$.

(a) Find the reduced mass for the relative motion of the tau (τ) lepton and the proton in the hydrogen tau atom.

(b) By applying the Bohr's quantization rule, find the first Bohr's radius a_0 and the energy levels E_n of the hydrogen tau atom.

Solution: (a) The reduced mass is

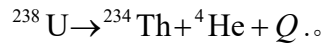
$$\mu = \frac{m_\tau m_p}{m_\tau + m_p}.$$

(b)

$$\begin{aligned} \mu \frac{v^2}{r} &= \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \Rightarrow (\mu v r)^2 = \frac{1}{4\pi\epsilon_0} \mu e^2 r \\ \Rightarrow (n\hbar)^2 &= \frac{1}{4\pi\epsilon_0} \mu e^2 r_n \Rightarrow r_n = \frac{4\pi\epsilon_0 \hbar^2}{\mu e^2} n^2 \\ \Rightarrow r_n &= \left(\frac{m_\tau + m_p}{m_\tau m_p} \right) \frac{4\pi\epsilon_0 \hbar^2}{e^2} n^2 \\ \Rightarrow a_0 &= \left(\frac{m_\tau + m_p}{m_\tau m_p} \right) \frac{4\pi\epsilon_0 \hbar^2}{e^2}. \end{aligned}$$

$$\begin{aligned} E &= \frac{1}{2} \mu v^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} = \frac{1}{2} \left(\mu \frac{v^2}{r} \right) r - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} \\ &= \frac{1}{2} \left(\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \right) r - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} = -\frac{1}{8\pi\epsilon_0} \frac{e^2}{r}, \\ E_n &= -\frac{1}{8\pi\epsilon_0} \frac{e^2}{r_n} = -\frac{1}{8\pi\epsilon_0} \frac{e^2}{a_0} \frac{1}{n^2}. \end{aligned}$$

2. (20 pts.) The α decay process of a ^{238}U (Uranium-238) nucleus is



It is known that the decay energy (Q -value) of the process is 4.27 MeV. Here we assume that mother nucleus ^{238}U is initially at rest.

(a) Find the kinetic energy of the emitting α particle (^4He (Helium-4) nucleus).

(b) Find the kinetic energy of the recoiling daughter ^{234}Th (Thorium-234) nucleus.

Solution: (a)

$$p_{\text{Th}} = p_{\alpha}, \quad E_{\text{Th}} = E_{\alpha}, \quad E_{\text{Th}} = \frac{p_{\text{Th}}^2}{2m_{\text{Th}}}, \quad E_{\alpha} = \frac{p_{\alpha}^2}{2m_{\alpha}}$$

$$\Rightarrow E_{\text{Th}} = \frac{m_{\alpha}}{m_{\text{Th}}} E_{\alpha}.$$

$$Q = E_{\text{Th}} + E_{\alpha}$$

$$\Rightarrow Q = \left(1 + \frac{m_{\alpha}}{m_{\text{Th}}}\right) E_{\alpha}$$

$$\Rightarrow E_{\alpha} = \frac{Q}{1 + \frac{m_{\alpha}}{m_{\text{Th}}}} = \frac{m_{\text{Th}}}{m_{\text{Th}} + m_{\alpha}} Q$$

$$\Rightarrow E_{\alpha} = \frac{234}{234 + 4} \times 4.26 \text{ MeV} = 4.188 \text{ MeV}.$$

(b) $E_{\text{Th}} = Q - E_{\alpha} = 4.26 \text{ MeV} - 4.188 \text{ MeV} = 0.072 \text{ MeV}.$

3. (20 pts.) For each of the following reactions, identify it as allowed or forbidden. If it is forbidden, state at least one conservation law that is violated. If it is allowed, state the main interaction the reaction is mediated.

	allowed/forbidden	violated conservation law/main interaction
(a) $p + p \rightarrow p + p + K^+ + K^-$	(a) allowed	strong interaction
(b) $p + K^- \rightarrow \Sigma^+ + n + \pi^+ + \pi^-$	(b) forbidden	The baryon number is not conserved.
(c) $\pi^+ + p \rightarrow K^+ + \Sigma^+$	(c) allowed	strong interaction
(d) $\bar{\nu}_{\mu} + p \rightarrow \mu^+ + n$	(d) allowed	weak interaction
(e) $p + p \rightarrow p + n + K^+$	(e) forbidden	The strangeness is not conserved. (It should be a strong interaction.)
(f) $K^+ \rightarrow \pi^+ + \pi^0$	(f) allowed	weak interaction
(g) $K^+ \rightarrow \mu^+ + \nu_{\mu}$	(g) allowed	weak interaction

The quark combinations of relevant hadrons:

$p - uud$ $n - udd$ $\Sigma^+ - uus$ $K^+ - u\bar{s}$ $K^- - s\bar{u}$ $\pi^+ - u\bar{d}$ $\pi^- - d\bar{u}$

4. (20 pts.) The electronic stationary wave function of a hydrogen atom at its ground state can be written in the form

$$\psi(r) = Ce^{-r/a_0}$$

where r is the radial distance in the spherical coordinates, a_0 the Bohr radius and C the normalization coefficient to be determined.

(a) Determine the normalization coefficient C in terms of a_0 .

(b) Find the corresponding radial probability density $w(r)$.

(c) Calculate the most probable radius r_{\max} and the average radius $\langle r \rangle$ of the electron.

Solution: (a) From

$$1 = \int_0^{\infty} |\psi(r)|^2 4\pi r^2 dr = |C|^2 4\pi \int_0^{\infty} r^2 e^{-2r/a_0} dr = |C|^2 4\pi \left(\frac{a_0}{2}\right)^3 \int_0^{\infty} u^2 e^{-u} du = |C|^2 4\pi \left(\frac{a_0}{2}\right)^3 \Gamma(3) = |C|^2 \pi a_0^3,$$

we obtain $C = 1/\sqrt{\pi a_0^3}$.

(b) $w(r)dr = |\psi(r)|^2 4\pi r^2 dr = \frac{4r^2}{a_0^3} e^{-2r/a_0} dr$, $w(r) = \frac{4r^2}{a_0^3} e^{-2r/a_0}$.

(c) From $\frac{dw}{dr} = \left[\frac{8r}{a_0^3} + \frac{4r^2}{a_0^3} \left(-\frac{2}{a_0} \right) \right] e^{-2r/a_0} = 0$, we have.

$$\frac{8r}{a_0^3} \left(1 - \frac{r}{a_0} \right) = 0 \Rightarrow r = 0, r = a_0.$$

It can be checked that the maximum probability is at $r_{\max} = a_0$. ($w(r=0) = 0$.)

$$\langle r \rangle = \int_0^{\infty} r w(r) dr = \frac{4}{a_0^3} \int_0^{\infty} r^3 e^{-2r/a_0} dr = \frac{4}{a_0^3} \left(\frac{a_0}{2}\right)^4 \int_0^{\infty} u^3 e^{-u} du = \frac{a_0}{4} \Gamma(4) = \frac{3}{2} a_0.$$

5. (20 pts.) A particle of mass m moves in a one-dimensional infinite potential well

$$U(x) = \begin{cases} 0, & 0 < x < a, \\ +\infty, & x \leq 0 \text{ or } x \geq a, \end{cases}$$

where $a > 0$ is the width of the well. There is a normalized wave function

$$\psi(x,t) = \begin{cases} \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) e^{-i\omega_1 t} + A \sqrt{\frac{2}{a}} \sin(k_2 x) e^{-i\omega_2 t}, & 0 < x < a, \\ 0, & x \leq 0 \text{ or } x \geq a, \end{cases}$$

where $k_1 = \pi/a$, $k_2 = 2\pi/a$, $\omega_1 = \hbar k_1^2 / (2m)$, $\omega_2 = \hbar k_2^2 / (2m)$ and A is a constant to be determined.

(a) Verify that the wave function satisfies the Schrodinger equation and the boundary conditions for the potential well.

(b) Determine the constant A .

(c) Find the corresponding probability density $\rho(x,t)$ of the particle.

(d) Find the average position $\langle x \rangle$ of the particle as the function of time t .

Solution: (a)

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \psi(x,t) &= i\hbar \frac{\partial}{\partial t} \left[\frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) \exp(-i\omega_1 t) + A \sqrt{\frac{2}{a}} \sin(k_2 x) \exp(-i\omega_2 t) \right] \\
&= \hbar \omega_1 \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) \exp(-i\omega_1 t) + \hbar \omega_2 A \sqrt{\frac{2}{a}} \sin(k_2 x) \exp(-i\omega_2 t). \\
-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x,t) &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left[\frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) \exp(-i\omega_1 t) + A \sqrt{\frac{2}{a}} \sin(k_2 x) \exp(-i\omega_2 t) \right] \\
&= \frac{\hbar^2 k_1^2}{2m} \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) \exp(-i\omega_1 t) + \frac{\hbar^2 k_2^2}{2m} A \sqrt{\frac{2}{a}} \sin(k_2 x) \exp(-i\omega_2 t) \\
&= \hbar \omega_1 \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) \exp(-i\omega_1 t) + \hbar \omega_2 A \sqrt{\frac{2}{a}} \sin(k_2 x) \exp(-i\omega_2 t). \\
\Rightarrow i\hbar \frac{\partial}{\partial t} \psi(x,t) &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x,t). \quad [3\text{pts}]
\end{aligned}$$

$$\psi(x,t)|_{x=0} = \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 \cdot 0) \exp(-i\omega_1 t) + A \sqrt{\frac{2}{a}} \sin(k_2 \cdot 0) \exp(-i\omega_2 t) = 0,$$

$$\begin{aligned}
\psi(x,t)|_{x=a} &= \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 a) \exp(-i\omega_1 t) + A \sqrt{\frac{2}{a}} \sin(k_2 a) \exp(-i\omega_2 t) \\
&= \frac{3}{5} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} a\right) \exp(-i\omega_1 t) + A \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a} a\right) \exp(-i\omega_2 t) \\
&= 0.
\end{aligned}$$

(b)

$$\begin{aligned}
1 &= \int_0^a |\psi(x,t)|^2 dx = \int_0^a \psi^*(x,t) \psi(x,t) dx \\
&= \int_0^a \left[\frac{9}{25} \frac{2}{a} \sin^2(k_1 x) + A^2 \frac{2}{a} \sin^2(k_2 x) + \frac{3}{5} \frac{2}{a} A \sin(k_1 x) \sin(k_2 x) [\exp(i(\omega_1 - \omega_2)t) + \exp(-i(\omega_1 - \omega_2)t)] \right] dx \\
&= \frac{9}{25} + A^2, A = \frac{4}{5}.
\end{aligned}$$

(c)

$$\begin{aligned}
\psi(x,t) &= \frac{3}{5} \sqrt{\frac{2}{a}} \sin(k_1 x) \exp(-i\omega_1 t) + \frac{4}{5} \sqrt{\frac{2}{a}} \sin(k_2 x) \exp(-i\omega_2 t), \\
\rho(x,t) &= \psi^*(x,t) \psi(x,t) \\
&= \frac{9}{25} \frac{2}{a} \sin^2(k_1 x) + \frac{16}{25} \frac{2}{a} \sin^2(k_2 x) + \frac{12}{25} \frac{2}{a} \sin(k_1 x) \sin(k_2 x) [\exp(i(\omega_1 - \omega_2)t) + \exp(-i(\omega_1 - \omega_2)t)] \\
&= \frac{18}{25a} \sin^2(k_1 x) + \frac{32}{25a} \sin^2(k_2 x) + \frac{48}{25a} \sin(k_1 x) \sin(k_2 x) \cos((\omega_1 - \omega_2)t).
\end{aligned}$$

$$(d) \quad \langle x \rangle = \int_0^a x \rho(x,t) dx = \frac{a}{2} - \frac{64a}{75\pi^2} \cos[(\omega_1 - \omega_2)t].$$

6.(20 pts.) Consider a gas of N extremely relativistic spin-1/2 fermions that are confined in a volume V . The energy-momentum relation of the fermions is $\varepsilon = pc$.

(a) Find the density of states (i.e., the number of quantum states per unit energy interval) of the system.

(b) Find the Fermi energy E_F of the system.

(c) Show that the total energy of the system at $T = 0$ K is $(3/4)NE_F$.

Solution: (a)

$$g_s \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} = 2V \frac{k^2 dk}{2\pi^2} = V \frac{p^2 dp}{\pi^2 \hbar^3} = \frac{V}{\pi^2 \hbar^3 c^3} \varepsilon^2 d\varepsilon = D(\varepsilon) d\varepsilon.$$

The density of states is $D(\varepsilon) = \frac{V}{\pi^2 \hbar^3 c^3} \varepsilon^2$.

(b) Form

$$N = \int_0^\infty \theta(\varepsilon - E_F) D(\varepsilon) d\varepsilon = \frac{V}{\pi^2 \hbar^3 c^3} \int_0^{E_F} \varepsilon^2 d\varepsilon = \frac{V}{3\pi^2 \hbar^3 c^3} E_F^3.$$

we obtain $E_F = c\hbar \left(\frac{3\pi^2 N}{V} \right)^{\frac{1}{3}}$.

$$\left[N = g_s \frac{4}{3} \frac{\pi k_F^3}{\left(\frac{2\pi}{L}\right)^3} = V \frac{k_F^3}{3\pi^3}, \quad k_F = \left(\frac{3\pi^2 N}{V} \right)^{\frac{1}{3}}, \quad E_F = c\hbar k_F = c\hbar \left(\frac{3\pi^2 N}{V} \right)^{\frac{1}{3}}. \right]$$

(c) The total energy is

$$E = \int_0^\infty \varepsilon \theta(\varepsilon - E_F) D(\varepsilon) d\varepsilon = \frac{V}{\pi^2 \hbar^3 c^3} \int_0^{E_F} \varepsilon^3 d\varepsilon = \frac{V}{4\pi^2 \hbar^3 c^3} E_F^4 = \frac{3}{4} \frac{V}{3\pi^2 \hbar^3 c^3} E_F^3 \cdot E_F = \frac{3}{4} NE_F.$$