1	2	3	4	5	6	Sum
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Name____________________________ Jan 11, 2019 Friday

SID_____________________________ School/Department______________________

University Physics

Final Examination

Kuang Yaming Honors School, Nanjing University

$$\Gamma(\nu)=\int_0^{+\infty} e^{-t}t^{\nu-1}dt\,,\ \ \Gamma(\nu+1)=\nu\Gamma(\nu),\ \ \Gamma(1)=1\,,\ \ \Gamma(1/2)=\sqrt{\pi}\,;\int_{-\infty}^{+\infty} e^{-x^2}dx=\sqrt{\pi}\,.$$

c	h	$\hbar$	e	m_e	m_n	u	N_A	k_B
3.0×10^8 m/s	6.63×10^{-34} J·s 1.24×10^3eV ·nm/c	1.05×10^{-34} J·s	1.6×10^{-19} C	9.11×10^{-31} kg 0.511MeV/c^2	939.565 MeV/c^2	931.494 MeV/c^2	6.02×10^{23} mol^{-1}	8.617×10^{-5} eV/K

Quantum numbers of three quarks:

Quark symbol	Charge number	Spin	Baryon number	Strangeness
u	2/3	1/2	1/3	0
d	−1/3	1/2	1/3	0
s	−1/3	1/2	1/3	−1

Select five out of the following six problems.

1.(20 pts.) In the experiment of photoelectric effect using iron as emitter, it is found that the threshold frequency of light is $\nu_C = 1.10\times10^{15}$ Hz.

(a) Find the work function ϕ of iron in the unit of eV.

(b) Find the stopping voltage V_0 for the photoelectrons emitted when light of wave length $\lambda = 250\text{nm}$ falls on the surface of iron.

Solution: (a)

$$\phi = h\nu_C = \frac{hc\nu_C}{c} = \frac{1.24\times10^3\,\text{eV}\cdot\text{nm}\cdot1.10\times10^{15}\,\text{s}^{-1}}{3.0\times10^8\,\text{ms}^{-1}} = 4.54\text{eV}\,.$$

(b)

$$eV_0 = K_{\max} = \frac{hc}{\lambda} - \phi = \frac{1.24\times10^3\,\text{eV}\cdot\text{nm}}{250\text{nm}} - 4.54\text{eV} = 0.52\text{eV}\,,\quad V_0 = 0.52\text{V}\,.$$

2. (20 pts.) In the beta decay of ${}^{24}_{11}\text{Na}$, the daughter nucleus is ${}^{24}_{12}\text{Mg}$ and an electron with kinetic energy of 2.15 MeV is observed. The atomic mass of the ${}^{24}_{11}\text{Na}$ is $m\left({}^{24}_{11}\text{Na}\right) = 23.990963\,\text{u}$ and the atomic mass of ${}^{24}_{12}\text{Mg}$ is $m\left({}^{24}_{12}\text{Mg}\right) = 23.985042\,\text{u}$.

(a) Write down the nuclear equation of the decay.

(b) Find the Q-value of the decay.

(c) What is the energy of the accompanying neutrino/antineutrino?

Solution: (a)

$$^{24}_{11}\mathrm{Na} \rightarrow ^{24}_{12}\mathrm{Mg} + \mathrm{e}^{-1} + \overline{\nu}_e$$

(b) $$Q = \left(m\left(^{24}_{11}\mathrm{Na}\right) - m\left(^{24}_{12}\mathrm{Mg}\right)\right)c^2 = \left(23.990963\,\mathrm{u} - 23.985042\mathrm{u}\right)c^2 = 5.52\,\mathrm{MeV}.$$

(c) Because the mass of the daughter nucleus is much larger than the masses of electron and antineutrino, the energy released in the decay is mostly carried away by the electron and the antineutrino. Thus

$$E_\nu = Q - K_e = 5.52\mathrm{eV} - 2.15\mathrm{eV} = 3.37\mathrm{eV}.$$

3. (20 pts.) For each of the following reactions, identify it is allowed or forbidden. If it is forbidden, state a conservation law that is violated.

(a) $\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \Lambda^0 + \mathrm{K}^+$ allowed

(b) $\mathrm{K}^0 \rightarrow \pi^+ + e^-$ forbidden The electron lepton number L_e is not conserved.

(c) $\mathrm{p} + \mathrm{n} \rightarrow \mathrm{p} + \Lambda^0$ forbidden It is strong interaction but the strangeness S is not conserved.

(d) $\Omega^- \rightarrow \Lambda^0 + \mathrm{K}^-$ allowed

(e) $\mathrm{n} + \nu_\mathrm{e} \rightarrow \mathrm{p} + e^-$ allowed

(f) $\Lambda^0 \rightarrow \mathrm{p} + \pi^-$ allowed

(g) $\mathrm{p} + \overline{\mathrm{p}} \rightarrow \Lambda^0 + \Lambda^0$ forbidden The baryon number B is not conserved.

The quark combinations of relevant hadrons:

$\mathrm{p} - \mathrm{uud}$ $\Lambda^0 - \mathrm{uds}$ $\Omega^- - \mathrm{sss}$ $\overline{\mathrm{p}} - \overline{\mathrm{u}}\overline{\mathrm{u}}\overline{\mathrm{d}}$ $\mathrm{K}^+ - \mathrm{u}\overline{\mathrm{s}}$ $\mathrm{K}^0 - \mathrm{d}\overline{\mathrm{s}}$ $\mathrm{K}^- - \mathrm{s}\overline{\mathrm{u}}$ $\pi^+ - \mathrm{u}\overline{\mathrm{d}}$ $\pi^- - \mathrm{d}\overline{\mathrm{u}}$

4. (20 pts.) An ideal Bose or Fermi gas is composed of particles whose energy-momentum relation is $\varepsilon = p^2/(2m)$. The distribution function, i.e., the average number of particle in a quantum state with energy ε, can be written as $f(\varepsilon) = 1/\left(e^{\beta(\varepsilon-\mu)} \pm 1\right)$ where $\beta = 1/k_B T$, μ is the chemical potential of the gas , and the positive and negative signs are for Fermi and Bose gas respectively. However, if $e^{-\beta\mu} >> 1$, the ± 1 in the denominator is negligible and the quantum distributions both reduce to the classical limit of Boltzmann distribution $f_C(\varepsilon) = e^{-\beta(\varepsilon-\mu)}$.

(1) Show that the density of states (DOS) of the gas can be expressed as

$$\rho(\varepsilon) = \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

where g_s is the degree of degeneracy due to the spin of the particle and V is the volume of the gas.

(2) Show that total number of particles N in the classical Boltzmann limit can be expressed as

$$N = g_s V e^{\beta\mu}\left(\frac{2\pi m}{\beta h^2}\right)^{3/2}.$$

(3) Furthermore, show that the condition for the classical limit (or the non-degenerate limit) $e^{-\beta\mu} >> 1$ is equivalent to the condition $n\lambda^3 << g_s$ where $n = N/V$ is the number density of particles and $\lambda = h/\sqrt{2\pi m k_B T}$ is the thermal de Broglie wavelength.

Solution: (a)

$$g_s \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} = g_s V \frac{1}{2\pi^2} k d\left(\frac{k^2}{2}\right) = g_s V \frac{1}{2\pi^2 \hbar^3} m\hbar k d\left(\frac{\hbar^2 k^2}{2m}\right).$$

$$= g_s V \frac{1}{2\pi^2 \hbar^3} m\sqrt{2m\varepsilon}\, d\varepsilon = \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon = \rho(\varepsilon) d\varepsilon$$

Thus $$\rho(\varepsilon) = \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}.$$

(b)

$$N = \int_0^\infty d\varepsilon \rho(\varepsilon) f_C(\varepsilon) = \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \varepsilon^{1/2} e^{-\beta(\varepsilon-\mu)}$$

$$= \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} e^{\beta\mu} \int_0^\infty d\varepsilon \varepsilon^{1/2} e^{-\beta\varepsilon} \overset{\beta\varepsilon = x}{=} \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2 \beta}\right)^{3/2} e^{\beta\mu} \int_0^\infty x^{1/2} e^{-x} dx$$

$$= \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2 \beta}\right)^{3/2} e^{\beta\mu}\Gamma\left(\frac{3}{2}\right) = \frac{g_s V}{4\pi^2}\left(\frac{2m}{\hbar^2 \beta}\right)^{3/2} e^{\beta\mu} \frac{\sqrt{\pi}}{2}.$$

$$= \frac{g_s V}{4\pi^2}\left(\frac{8\pi^2 m}{h^2 \beta}\right)^{3/2} e^{\beta\mu} \frac{\sqrt{\pi}}{2} = g_s V e^{\beta\mu}\left(\frac{2\pi m}{\beta h^2}\right)^{3/2}$$

(c) $$e^{-\beta\mu} = g_s \frac{V}{N}\left(\frac{2\pi m}{\beta h^2}\right)^{3/2} >> 1 \Rightarrow \frac{N}{V}\left(\frac{h^2 \beta}{2\pi m}\right)^{3/2} >> g_s \Rightarrow n\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} >> g_s$$

$$\Rightarrow n\lambda^3 >> g_s.$$

5. (20 pts.) An electron of mass m moves in a one-dimensional symmetric infinite potential well

$$U(x) = \begin{cases} 0, & |x| \le a/2, \\ +\infty, & |x > a/2,| \end{cases}$$

where $a > 0$ is the width of the well.

(a) Write down the stationary Schrödinger equation of the electron.

(b) Assuming that the solutions of the stationary Schrödinger equation take the form

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ika}, & |x| \le a/2 \\ 0, & \text{elsewhere} \end{cases}, \quad k > 0, \quad A > 0,$$

determine the possible values of wave vector k by applying the proper boundary conditions.

(c) Using the results of (b) to find the energy levels of the electron in the potential well.

(d) Determine the coefficients A and B in (b) by the normalization of the wave functions.

Solution: (a) $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)$.

(b) $\psi(x = -a/2) = 0$: $\quad Ae^{-i\frac{ka}{2}} + Be^{i\frac{ka}{2}} = 0$;

$\psi(x = a/2) = 0$: $\quad Ae^{i\frac{ka}{2}} + Be^{-i\frac{ka}{2}} = 0$.

The nontrivial solution of A and B requires

$$\begin{vmatrix} e^{-i\frac{ka}{2}} & e^{i\frac{ka}{2}} \\ e^{-i\frac{ka}{2}} & e^{-i\frac{ka}{2}} \end{vmatrix} = e^{-ika} - e^{ika} = 0 \quad \text{or} \quad e^{2ika} = 0.$$

Thus $\quad 2ka = 2n\pi$ or $k_n = \frac{n\pi}{a}$ where $n = 1,2,3,\cdots$. (

(c) $\psi_n(x) = \begin{cases} Ae^{ik_n x} + Be^{-ik_n a}, & |x| \le a/2 \\ 0, & \text{elsewhere} \end{cases}$. For $|x| \le \frac{a}{2}$, we have

$$\hat{H}\psi_n(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(Ae^{ik_n x} + Be^{-ik_n a}\right) = \frac{\hbar^2 k_n^2}{2m}\left(Ae^{ik_n x} + Be^{-ik_n a}\right) = \frac{\hbar^2 k_n^2}{2m}\psi_n(x).$$

Thus $E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

(d) From $Ae^{-i\frac{k_n a}{2}} + Be^{i\frac{k_n a}{2}} = 0$, we have $\frac{B}{A} = e^{-ik_n a} = e^{-in\pi} = (-1)^n$.

Thus, for $|x| \le \frac{a}{2}$, we have $\psi_n(x) = A\left[e^{ik_n x} + (-1)^n e^{-ik_n a}\right] = \begin{cases} 2A\cos(k_n x), & n = 2,4,6\cdots \\ 2iA\sin(k_n x), & n = 1,3,5\cdots \end{cases}$.

From $1 = \int_{-a/2}^{a/2} |\psi_n(x)|^2 dx = 2A^2 a$, we obtain $A = \sqrt{\frac{1}{2a}}$, $\quad B = (-1)^n\sqrt{\frac{1}{2a}}$.

6.(20 pts.) The Hamiltonian operator of a one-dimensional harmonic oscillator is $\hat{H} = \hat{p}^2/(2m) + (1/2)m\omega^2 x^2$ where m is the mass of the particle, $\hat{p} = -i\hbar(d/dx)$ is the momentum operator and ω is the natural circular frequency of the oscillator. The normalized ground state wave function of the oscillator is found to be

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{\alpha^2 x^2}{2}}, \quad \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}.$$

(a) Find the expectation values of position coordinator $\langle x \rangle$ and momentum $\langle p \rangle$ of the particle in the ground state.

(b) Find the uncertainties of position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and momentum $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, and show that the Heisenberg's uncertainty principle is satisfied.

(c) Find the expectation values of kinetic energy and potential energy of the oscillator in its ground state. Comment on their non-zero values and relative magnitude of two quantities.

Solution: (a)

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_0^*(x) x \psi_0(x) dx = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-\alpha^2 x^2} dx = 0,$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi_0^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi_0(x) dx = \frac{\alpha}{\sqrt{\pi}} \frac{\hbar}{i} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2 x^2}{2}} \frac{d}{dx}\left(e^{-\frac{\alpha^2 x^2}{2}}\right) dx = \frac{\alpha}{\sqrt{\pi}} \frac{\hbar}{i} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2 x^2}{2}} \left(-\alpha^2 x e^{-\frac{\alpha^2 x^2}{2}}\right) dx$$

$$= 0.$$

(b)

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x) x^2 \psi_0(x) dx = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx \overset{u=\alpha x}{=} \frac{\alpha}{\sqrt{\pi}} \frac{1}{\alpha^3} \int_{-\infty}^{\infty} u^2 e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}\alpha^2} \int_0^{\infty} u^2 e^{-u^2} du \overset{t=u^2}{=} \frac{2}{\sqrt{\pi}\alpha^2} \int_0^{\infty} t e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{\sqrt{\pi}\alpha^2} \int_0^{\infty} t^{1/2} e^{-t} dt$$

$$= \frac{1}{\sqrt{\pi}\alpha^2} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\pi}\alpha^2} \frac{\sqrt{\pi}}{2} = \frac{1}{2\alpha^2} = \frac{\hbar}{2m\omega}.$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x) \left(\frac{\hbar}{i} \frac{d}{dx}\right)^2 \psi_{0^2}(x) dx = -\frac{\alpha\hbar^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2 x^2}{2}} \frac{d^2}{dx^2}\left(e^{-\frac{\alpha^2 x^2}{2}}\right) dx$$

$$= -\frac{\alpha\hbar^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2 x^2}{2}} \left(-\alpha^2 + \alpha^4 x^2\right) e^{-\frac{\alpha^2 x^2}{2}} dx. = \frac{\alpha^3\hbar^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} dx - \frac{\alpha^5\hbar^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx$$

$$\overset{u=\alpha x}{=} \frac{\alpha^2\hbar^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du - \frac{\alpha^2\hbar^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du = \alpha^2\hbar^2 - \frac{\alpha^2\hbar^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{\alpha^2\hbar^2}{2} = \frac{m\omega\hbar}{2}.$$

Thus $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$, $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\omega\hbar}{2}}$, and $\Delta x \Delta p = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}$.

Therefore, the Heisenberg's uncertainty principle is satisfied.

(c) $\langle T \rangle = \left\langle \frac{p^2}{2m} \right\rangle = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar\omega}{4}$, $\langle V \rangle = \left\langle \frac{m\omega^2 x^2}{2} \right\rangle = \frac{m\omega^2 \langle x^2 \rangle}{2} = \frac{\hbar\omega}{4}$.

The fact that lowest possible energy of the oscillator cannot be zero is due to the Heisenberg principle of uncertainty: the particle cannot stay at the state with $x = 0$ and $p = 0$ simultaneously. The equal quantities of expectation values kinetic energy and potential energy is quite similar to that in a classical harmonic oscillator whose average kinetic energy and potential energy during a period is of the same value.