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Name _____

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SID _____

School/Department _____

University Physics

Final Examination

Kuang Yaming Honors School, Nanjing University

$$\Gamma(\nu) = \int_0^{+\infty} e^{-t} t^{\nu-1} dt, \quad \Gamma(\nu + 1) = \nu \Gamma(\nu), \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}; \quad \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

c	h	ħ	e	m _e	m _n	u	N _A	k _B
3.0×10 ⁸ m/s	6.63×10 ⁻³⁴ J·s 1.24×10 ³ eV·nm/c	1.05×10 ⁻³⁴ J·s	1.6×10 ⁻¹⁹ C	9.11×10 ⁻³¹ kg 0.511MeV/c ²	939.565 MeV/c ²	931.494 MeV/c ²	6.02×10 ²³ mol ⁻¹	8.617×10 ⁻⁵ eV/K

Quantum numbers of three quarks:

Quark symbol	Charge number	Spin	Baryon number	Strangeness
u	2/3	1/2	1/3	0
d	-1/3	1/2	1/3	0
s	-1/3	1/2	1/3	-1

Select five out of the following six problems.

1.(20 pts.) An X-ray photon of wavelength $\lambda_0 = 0.0300$ nm makes a collision with an electron that is initially at rest and undergoes a 90° Compton scattering.

(a) Find the wavelength of scattered photon.

(b) Find the kinetic energy of the electron after the scattering.

Solution: (a)

$$\Delta\lambda = \frac{h}{m_e c} (1 - \cos\phi) = \frac{hc}{m_e c^2} (1 - \cos 90^\circ) = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{0.511 \text{ MeV}} = 2.42 \times 10^{-3} \text{ nm}.$$

$$\lambda = \lambda_0 + \Delta\lambda = 0.0324 \text{ nm}.$$

(b) From energy conservation

$$\frac{hc}{\lambda_0} + m_e c^2 = \frac{hc}{\lambda} + m_e c^2 + K.$$

$$K = \frac{hc}{\lambda_0} - \frac{hc}{\lambda} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{0.0300 \text{ nm}} - \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{0.0324 \text{ nm}} = 3.06 \times 10^3 \text{ eV}.$$

2. (20 pts.) The half-life of neutron is $t_{1/2} = 10.2$ min. A beam of neutrons of kinetic energy 5 eV is emitted in a direction.

(a) What is the de Broglie wavelength of the neutrons?

(b) Over what distance will the intensity of the neutron beam be reduced to the half of its original value?

(c) What is the average distance a neutron in the beam travels before it decays?

Solution: (a) Since $5\text{eV} \ll m_n c^2$, the non-relativistic approach can be used.

$$\lambda = \frac{h}{\sqrt{2m_n E_K}} = \frac{hc}{\sqrt{2m_n c^2 E_K}} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{nm}}{\sqrt{2 \times 9.40 \times 10^8 \text{ eV} \times 5\text{eV}}} = 1.28 \times 10^{-2} \text{ nm}.$$

$$(b) \quad v = \sqrt{\frac{2E_K}{m_n}} = \sqrt{\frac{2E_K}{m_n c^2}} c = \sqrt{\frac{10\text{eV}}{940 \times 10^6 \text{eV}}} = 1.06 \times 10^{-4} c = 3.19 \times 10^4 \text{ m/s}.$$

$$d = vt_{1/2} = 1.95 \times 10^7 \text{ m}.$$

$$(c) \quad N = N_0 e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{t_{1/2}}, \quad -dN = \lambda N_0 e^{-\lambda t} dt.$$

$$\bar{t} = \frac{\int_0^{+\infty} t(-dN)}{\int_0^{+\infty} (-dN)} = \frac{\int_0^{+\infty} t e^{-\lambda t} dt}{\int_0^{+\infty} e^{-\lambda t} dt} = \frac{\lambda^{-2} \int_0^{+\infty} u e^{-u} du}{\lambda^{-1} \int_0^{+\infty} e^{-u} du} = \frac{1}{\lambda} \frac{\Gamma(2)}{\Gamma(1)} = \frac{1}{\lambda} = \frac{t_{1/2}}{\ln 2} = 14.7 \text{ min}.$$

$$\bar{d} = v\bar{t} = 2.81 \times 10^7 \text{ m}.$$

3. (20 pts.) A particle of mass m is moving in a one-dimensional simple harmonic potential and its natural circular frequency is ω . The particle is in a stationary quantum state of the normalized wave function

$$\psi(x) = C e^{-\frac{1}{2}\alpha^2 x^2}$$

where $\alpha = \sqrt{m\omega/\hbar}$ and C is a positive constant.

(a) Write the Hamiltonian operator of the particle. Determine the constant C .

(b) Is the wave function an eigenfunction of the Hamiltonian?

(c) Find the averages $\langle xp \rangle$ and $\langle px \rangle$ in the quantum state, where x and p are position and momentum of the particle.

$$\text{Solution: (a)} \quad \hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2.$$

$$1 = \int \psi^*(x) \psi(x) dx = |C|^2 \int_{-\infty}^{+\infty} e^{-\alpha^2 x^2} dx = |C|^2 \int_{-\infty}^{+\infty} e^{-u^2} du = \frac{|C|^2}{\alpha} \int_{-\infty}^{+\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{\alpha} |C|^2, \quad C = \sqrt{\frac{\alpha}{\sqrt{\pi}}}.$$

(b)

$$\begin{aligned} \hat{H}\psi(x) &= C \hat{H} e^{-\frac{1}{2}\alpha^2 x^2} = C \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left(e^{-\frac{1}{2}\alpha^2 x^2} \right) + \frac{1}{2} m \omega^2 x^2 e^{-\frac{1}{2}\alpha^2 x^2} \right] = C \left[-\frac{\hbar^2}{2m} \frac{d}{dx} \left(-\alpha^2 x e^{-\frac{1}{2}\alpha^2 x^2} \right) + \frac{1}{2} m \omega^2 x^2 e^{-\frac{1}{2}\alpha^2 x^2} \right] \\ &= C \left[\frac{\hbar^2 \alpha^2}{2m} e^{-\frac{1}{2}\alpha^2 x^2} - \frac{\hbar^2}{2m} \alpha^4 x^2 e^{-\frac{1}{2}\alpha^2 x^2} + \frac{1}{2} m \omega^2 x^2 e^{-\frac{1}{2}\alpha^2 x^2} \right] = \frac{\hbar\omega}{2} C e^{-\frac{1}{2}\alpha^2 x^2} = \frac{\hbar\omega}{2} \psi(x). \end{aligned}$$

Therefore, $\psi(x)$ is an eigenfunction of \hat{H} .

(c)

$$\begin{aligned} \langle xp \rangle &= \int \psi^*(x) x \hat{p} \psi(x) dx = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha^2 x^2} x \frac{\hbar}{i} \frac{d}{dx} \left(e^{-\frac{1}{2}\alpha^2 x^2} \right) dx = -\frac{\hbar}{i} \frac{\alpha^3}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha^2 x^2} x^2 e^{-\frac{1}{2}\alpha^2 x^2} dx \\ &= -\frac{\hbar}{i} \frac{\alpha^3}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\alpha^2 x^2} dx = -\frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u^2 e^{-u^2} du = -\frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{\frac{1}{2}} e^{-t} dt = -\frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{i\hbar}{2}. \end{aligned}$$

$$\begin{aligned} \langle px \rangle &= \int \psi^*(x) \hat{p} x \psi(x) dx = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha^2 x^2} \frac{\hbar}{i} \frac{d}{dx} \left(x e^{-\frac{1}{2}\alpha^2 x^2} \right) dx \\ &= \frac{\hbar}{i} \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha^2 x^2} (1 - \alpha^2 x^2) e^{-\frac{1}{2}\alpha^2 x^2} dx = \frac{\hbar}{i} \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha^2 x^2} dx - \frac{\hbar}{i} \frac{\alpha^3}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\alpha^2 x^2} dx \\ &= \frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} du - \frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u^2 e^{-u^2} du = \frac{\hbar}{i} - \frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{\frac{1}{2}} e^{-t} dt = \frac{\hbar}{i} - \frac{\hbar}{i} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{\hbar}{2i}. \end{aligned}$$

4. (20 pts.) A neutron star that is composed almost entirely of N neutrons can be approximately treated as ideal Fermi gas of uniform density in a sphere of radius R at temperature $T = 0$.

(a) Find the Fermi energy of the neutron gas in terms of neutron number N , radius of the star R and the mass of neutron m_n .

(b) Find the total kinetic energy of the neutron gas E_K .

(c) The gravitational energy of the neutron star can be written as

$$E_G = -\frac{3}{5} \frac{GM^2}{R} = -\frac{3}{5} \frac{GN^2 m_n^2}{R}.$$

The star is in equilibrium when the inward gravitational force is just opposed by the repulsion due to the Pauli principle, that is, when the total energy $E = E_K + E_G$ reaches a minimum value. Show that the radius of the neutron star is given by

$$R = \frac{\hbar^2}{GN^{1/3} m_n^3} \left(\frac{9\pi}{4} \right)^{2/3} = \frac{h^2}{GN^{1/3} m_n^3} \left(\frac{9}{32\pi^2} \right)^{2/3}.$$

Solution: (a)

$$N = 2 \times \frac{4}{3} \pi k_F^3 \left(\frac{2\pi}{L} \right)^3 = N = V \frac{k_F^3}{3\pi^2}, \quad k_F^3 = 3\pi^2 \frac{N}{V} = \frac{3\pi^2 N}{\frac{4}{3}\pi R^3} = \frac{9\pi N}{4R^3}, \quad k_F = \left(\frac{9\pi N}{4} \right)^{1/3} \frac{1}{R}.$$

$$E_F = \frac{\hbar^2}{2m_n R^2} \left(\frac{9\pi N}{4} \right)^{2/3}$$

(b)

$$\bar{E} = \frac{\int_0^{+\infty} dE E g(E) \theta(E - E_F)}{\int_0^{+\infty} dE g(E) \theta(E - E_F)} = \frac{\int_0^{+\infty} dE E \sqrt{E} \theta(E - E_F)}{\int_0^{+\infty} dE \sqrt{E} \theta(E - E_F)} = \frac{\int_0^{E_F} E^2 dE}{\int_0^{E_F} E^{1/2} dE} = \frac{3}{5} E_F, \quad E_K = N \bar{E} = \frac{3}{5} N E_F.$$

$$(c) \quad E = E_K + E_G = \frac{3\hbar^2 N}{10m_n} \left(\frac{9\pi N}{4} \right)^{2/3} \frac{1}{R^2} - \frac{3}{5} \frac{GN^2 m_n^2}{R}.$$

$$\frac{dE}{dR} = -\frac{3\hbar^2 N}{5m_n} \left(\frac{9\pi N}{4} \right)^{2/3} \frac{1}{R^3} + \frac{3}{5} \frac{GN^2 m_n^2}{R^2} = 0, \quad R = \frac{\hbar^2}{GN^{1/3} m_n^3} \left(\frac{9\pi}{4} \right)^{2/3} = \frac{h^2}{GN^{1/3} m_n^3} \left(\frac{9}{32\pi^2} \right)^{2/3}.$$

5. (20 pts.) Consider a one-dimensional potential step

$$U(x) = U_0 \theta(x) = \begin{cases} 0, & x < 0, \\ U_0, & x \geq 0, \end{cases} \quad (U_0 > 0)$$

and incident particles of mass m and energy $E < U_0$ coming from left ($x < 0$) to collide with the step. The solutions of the wave equations can be written as

$$\psi(x) = A e^{ikx} + B e^{-ikx} \quad \text{for } x < 0;$$

$$\psi(x) = C e^{-\kappa x} \quad \text{for } x \geq 0$$

where $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(U_0 - E)}/\hbar$.

(a) What is the meaning of the each term of the solutions above?

(b) By applying the boundary conditions, determine the ratios B/A and C/A .

(c) Find the reflection coefficient R and the transmission coefficient T of the particles.

Solution: (a) Ae^{ikx} represents the incident wave, B^{-ikx} the reflected wave and $Ce^{-\kappa x}$ the transmitted wave.

(b) From $\psi(0^-) = \psi(0^+)$, we have $A + B = C$ or $1 + \frac{B}{A} = \frac{C}{A}$

From $\psi'(0^-) = \psi'(0^+)$, we have $ik(A - B) = -\kappa C$ or $1 - \frac{B}{A} = i \frac{\kappa C}{k A}$

Thus

$$\frac{C}{A} = \frac{2}{1 + i \frac{\kappa}{k}}, \quad \frac{B}{A} = \frac{C}{A} - 1 = \frac{2}{1 + i \frac{\kappa}{k}} - 1 = \frac{1 - i \frac{\kappa}{k}}{1 + i \frac{\kappa}{k}}$$

(c) $R = \frac{|B|^2}{|A|^2} = \frac{|B|^2}{|A|^2} = \left| \frac{1 - i \frac{\kappa}{k}}{1 + i \frac{\kappa}{k}} \right|^2 = 1, \quad T = 1 - R = 0.$

6.(20 pts.) (a) For the hadrons and their quark combinations listed in the first column of the following table, classify them as baryon and meson, find the charge number, baryon number and strangeness of them and write the results into the table.

Particle and quark combination	Classification	Baryon number	Charge number	Strangeness
$p - uud$	baryon	1	+1	0
$\Lambda^0 - uds$	baryon	1	0	-1
$K^- - s\bar{u}$	meson	0	-1	-1
$\pi^+ - u\bar{d}$	meson	0	+1	0
$\Omega^- - sss$	baryon	1	-1	-3
$K^0 - d\bar{s}$	meson	0	0	+1
$n - udd$	baryon	1	0	0
$K^+ - u\bar{s}$	meson	0	+1	+1

(b) Analyze following reactions. Identify them as "allowed" or "forbidden". If the reaction is allowed, indicate what interaction it is mainly mediated. If the reaction is forbidden, name the conservation law(s) violated.

Reactions or decays	Allowed/Forbidden	Main interaction/violated conservation law(s)
$p + p \rightarrow p + \pi^+ + \Lambda^0 + K^0$	allowed	strong interaction
$K^- + p \rightarrow n + \Lambda^0$	forbidden	conservation of baryon number
$\nu_e + p \rightarrow n + e^+$	forbidden	conservation of (electron) lepton number
$\Omega^- \rightarrow \Lambda^0 + K^-$	allowed	weak interaction
$p + p \rightarrow p + n + K^+$	forbidden	conservation of strangeness (for strong interaction)