1	2	3	4	5	6	Sum
/20	/20	/20	/20	/20	/20	/100

Name____________________________ Jan 3, 2017 Tuesday

SID______________________________ School/Department____________________

University Physics

Final Examination

Kuang Yaming Honors School, Nanjing University

$$\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt,\ \Gamma(\nu+1)=\nu\Gamma(\nu),\ \Gamma(1)=1,\ \Gamma(1/2)=\sqrt{\pi}.$$

c	h	$\hbar$	e	m_e	u	N_A	k_B
3.0×10^8 m/s	6.63×10^{-34} J·s 1.24×10^3eV ·nm/c	1.05×10^{-34} J·s	1.6×10^{-19} C	9.11×10^{-31} kg 0.511MeV/c^2	931.494MeV/c^2	6.02×10^{23} mol^{-1}	8.617×10^{-5}eV/K

The quark combinations of some hadrons: $\mathrm{p}-\mathrm{uud}$ $\mathrm{n}-\mathrm{udd}$ $\Lambda^0-\mathrm{uds}$ $\pi^+-\mathrm{u\bar{d}}$ $\pi^- -\mathrm{d\bar{u}}$ $\mathrm{K}^+-\mathrm{u\bar{s}}$ $\mathrm{K}^0-\mathrm{d\bar{s}}$

Quantum numbers of three quarks:

Quark symbols	Charge numbers	Spin	Baryon numbers	Strangeness
u	2/3	1/2	1/3	0
d	−1/3	1/2	1/3	0
s	−1/3	1/2	1/3	−1

Select five out of the following six problems.

1.(20 pts.) A ^{57}Fe nucleus in the first excited state decays to its ground state with the emission of a γ photon of 14.4 keV in a mean lifetime of 141 ns.

(a) Estimate the nuclear energy level width ΔE of the first excited state.

(b) What is the recoil kinetic energy E_K of an initially stationary ^{57}Fe nucleus when it emits a γ photon of 14.4 keV?

Solution:(a) By using the Heisenberg uncertainty relation $\Delta E\Delta t\sim\hbar$,we have

$$\Delta E\sim\frac{\hbar}{\Delta t}=\frac{hc}{2\pi c\Delta t}=\frac{1240\mathrm{eV}\cdot\mathrm{nm}}{6.28\times3.0\times10^8\,\mathrm{m/s}\times1.41\times10^{-8}\,\mathrm{s}}=4.67\times10^{-9}\,\mathrm{eV}.$$

(The results such as $2.34\times10^{-9}\,\mathrm{eV}$ or $2.93\times10^{-8}\,\mathrm{eV}$ are also acceptable.)

(b) The conservation of momentum gives

$$p_{Fe}=p_\gamma=\frac{E_\gamma}{c}.$$

Thus

$$E_K=\frac{p_{Fe}^2}{2m_{Fe}}=\frac{E_\gamma^2}{2m_{Fe}c^2}=\frac{(14.4\mathrm{keV})^2}{2\times57\mathrm{u}\times931.5\mathrm{MeV/u}}=1.95\times10^{-3}\,\mathrm{eV}.$$

2. (20 pts.) The plutonium ^{239}Pu is a by-product in nuclear reactor. It is radioactive, decaying by alpha decay with a half-life of 2.4×10^4 a (1 a = $\pi\times10^7$ s). Plutonium is also one of the most toxic chemicals known and its lethal dose to human is as little as 2 mg.

(a) How many nuclei does it have in the chemically lethal dose of ^{239}Pu?

(b) What is the decay rate (or activity) R of the amount of ^{239}Pu nuclei in (a) exactly after a time period of 7.2×10^4 a?

Solution: (a)

$$N_0=\frac{2.0\times10^{-3}\,\mathrm{g}}{239\mathrm{g/mol}}\times6.022\times10^{23}\,\mathrm{mol}^{-1}=5.04\times10^{18}.$$

$$R(t)=\lambda N_0 e^{-\lambda t}=\frac{\ln 2}{t_{1/2}}N_0 e^{-\frac{t\ln 2}{t_{1/2}}}=\frac{\ln 2}{t_{1/2}}\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$
$$=\frac{0.693}{2.4\times10^4\,\mathrm{a}\times\pi\times10^7\,\mathrm{s/a}}\times5.04\times10^{18}\times\left(\frac{1}{2}\right)^3$$
$$=5.8\times10^5\,\mathrm{s}^{-1}.$$

3.(20 pts.) (a) For the particles in the first column of the following table, classify the particles as lepton, baryon and meson, find the charge number, lepton number, baryon number and strangeness of them and write the results into the table.

Particles	Classification lepton/baryon/meson	Lepton number			Baryon number	Charge number	Strangeness
		L_e	L_μ	L_τ			
e^+	lepton	-1	0	0	0	1	N/A
K^0	meson	0	0	0	0	0	1
Λ^0	baryon	0	0	0	1	0	-1
ν_μ	lepton	0	1	0	0	0	N/A
p	baryon	0	0	0	1	1	0
π^-	meson	0	0	0	0	-1	0

(b) For the following reactions or decays, indicate if they are allowed or forbidden. Where it is forbidden, give a reason.

Reactions or decays	Allowed/Forbidden	Reason
$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$	allowed	
$p + p \to p + n + \pi^+$	allowed	
$K^0 \to \pi^- + \mu^+ + \nu_\mu$	allowed	
$n \to e^+ + e + \nu_e$	forbidden	The baryon number and e-lepton number are not conserved.
$K^- + p \to \Lambda^0 + K^0$	forbidden	It is collision between hadrons and is due to strong interaction but the strangeness is not conserved.

4. (20 pts.) A particle of mass *m* is moving in a one-dimensional simple harmonic potential and its natural circular frequency is ω. The particle is in a stationary quantum state of the normalized wave function

$$\psi(x)=Cx\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}$$

where $\alpha=\sqrt{m\omega/\hbar}$ and C is a positive constant to be determined.

(a) Determine the constant C.

(b) Find the most probable position(s) x_{mp} of the particle.

(c) Find the probability flux density J of the particle.

(Hint: The probability flux density is given by $\boldsymbol{J}=\frac{\hbar}{2im}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)$ where ψ is a normalized wave function.)

Solution: (a)

$$1=\int_{-\infty}^{\infty}|\psi|^2dx=|C|^2\int_{-\infty}^{\infty}x^2e^{-\alpha^2x^2}dx=2|C|^2\int_0^{\infty}x^2e^{-\alpha^2x^2}dx=\frac{2|C|^2}{\alpha^3}\int_0^{\infty}u^2e^{-u^2}du$$
$$=\frac{|C|^2}{\alpha^3}\int_0^{\infty}t^{\frac{1}{2}}e^{-t}dt=\frac{|C|^2}{\alpha^3}\Gamma\left(\frac{3}{2}\right)=\frac{|C|^2}{\alpha^3}\frac{\sqrt{\pi}}{2}.$$

Thus

$$C=\sqrt{2}\alpha\left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}}.$$

(b) The probability density is

$$\rho(x)=|\psi(x)|^2=\frac{2\alpha^3}{\sqrt{\pi}}x^2e^{-\alpha^2x^2}.$$

From $\frac{d\rho(x)}{dx}=0$, or $\frac{d\left(x^2e^{-\alpha^2x^2}\right)}{dx}=2xe^{-\alpha^2x^2}-2\alpha^2x^3e^{-\alpha^2x^2}=0,$ we have

$$x\left(1-\alpha^2x^2\right)=0.$$

Solving the equation gives $x=0$ (minimum) and $x=\pm\frac{1}{\alpha}$ (maxima).

Thus $$x_{mp}=\pm\frac{1}{\alpha}.$$

(b) For the 1D case, the probability current density can be written as

$$J=\frac{\hbar}{2im}\left(\psi^*\frac{d}{dx}\psi-\psi\frac{d}{dx}\psi^*\right).$$

Since in the present case, $\psi^\times=\psi$ (the wave function is real),

$$J=\frac{\hbar}{2im}\left(\psi^*\frac{d}{dx}\psi-\psi\frac{d}{dx}\psi^*\right)=\frac{\hbar}{2im}\left(\psi\frac{d}{dx}\psi-\psi\frac{d}{dx}\psi\right)=0.$$

5. (20 pts.) A particle of mass *m* moves in a one-dimensional interval [0, *a*] on the x-axis and the potential energy of the particle is identically zero within the interval. The wave function of the particle is assumed to satisfy the periodic boundary condition $\psi(0)=\psi(a)$.

(a) Show that the following states are eigenfunctions of both the momentum $\hat{p}$ and Hamiltonian $\hat{H}$.

$$\psi_n(x)=\begin{cases}\frac{1}{\sqrt{a}}e^{ik_nx}, & 0\le x\le a\\ 0, & \text{elsewhere,}\end{cases}\quad \text{where } k_n=\frac{2\pi n}{a},\ n=0,\pm1,\pm2,\cdots.$$

(b)Show that the eigenfunctions in (a) are orthonormalized, i.e. ,

$$\int_0^a\psi_m^*(x)\psi_n(x)dx=\delta_{mn},\quad \text{where } \delta_{mn}=\begin{cases}1, & m=n,\\ 0, & m\ne n\end{cases}\ \text{is the Kronecker delta.}$$

(c) If the particle is in a quantum state that is a linear combination of the eigenfunctions in (a), say

$$\psi(x)=\frac{1}{2}\psi_{-1}(x)+\frac{1}{2}\psi_1(x)+c\psi_5(x)$$

where c is a positive constant and $\psi(x)$ is normalized, find the constant c.

(d) Find the average momentum $\langle p\rangle$ and the uncertainty of momentum Δp if the particle is in the quantum state given in (c). .

Solution: (a) The Hamiltonian (operator) is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \qquad (0 \le x \le a) \ ,$$

And the momentum operator is $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$.

Therefore,

$$\hat{H}\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right) = \frac{\hbar^2 k_n^2}{2m}\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right) = E_n\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right);$$

$$\hat{p}\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right) = \frac{\hbar}{i}\frac{d}{dx}\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right) = \hbar k_n\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right) = p_n\left(\frac{1}{\sqrt{a}}e^{ik_n x}\right).$$

They are eigenfunctions of both the momentum operator $\hat{p}$ and Hamiltonian $\hat{H}$.

(b)

$$\int \psi_m^*(x)\psi_n(x)dx = \frac{1}{a}\int_0^a e^{i(k_n - k_m)x}dx = \frac{1}{a}\int_0^a e^{i\frac{2(n-m)\pi x}{a}}dx \, .$$

When $n = m$,

$$\int \psi_m^*(x)\psi_n(x)dx = \frac{1}{a}\int_0^a dx = 1 \, ;$$

and when $n \neq m$,

$$\int \psi_m^*(x)\psi_n(x)dx = \frac{1}{a}\int_0^a e^{i\frac{2(n-m)\pi x}{a}}dx = \frac{1}{i2(n-m)\pi}e^{i\frac{2(n-m)\pi x}{a}}\bigg|_0^a = 0 \, .$$

Thus $\int \psi_m^*(x)\psi_n(x)dx = \delta_{mn}$.

(c)

$$\begin{aligned} 1 = \int \psi^*(x)\psi(x)dx &= \int \left(\frac{1}{2}\psi_{-1}^*(x) + \frac{1}{2}\psi_1^*(x) + c^*\psi_5^*(x)\right)\left(\frac{1}{2}\psi_{-1}(x) + \frac{1}{2}\psi_1(x) + c\psi_5(x)\right)dx \\ &= \frac{1}{4}\int \psi_{-1}^*(x)\psi_{-1}(x)dx + \frac{1}{4}\int \psi_1^*(x)\psi_1(x)dx + |c|^2\int \psi_5^*(x)\psi_5(x)dx = \frac{1}{2} + |c|^2 . \end{aligned}$$

Thus $c = \frac{\sqrt{2}}{2}$.

$$\begin{aligned} \langle p \rangle &= \int \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}\psi(x)dx = \int \left(\frac{1}{2}\psi_{-1}^*(x) + \frac{1}{2}\psi_1^*(x) + \frac{\sqrt{2}}{2}\psi_5^*(x)\right)\frac{\hbar}{i}\frac{d}{dx}\left(\frac{1}{2}\psi_{-1}(x) + \frac{1}{2}\psi_1(x) + \frac{\sqrt{2}}{2}\psi_5(x)\right)dx \\ &= \int \left(\frac{1}{2}\psi_{-1}^*(x) + \frac{1}{2}\psi_1^*(x) + \frac{\sqrt{2}}{2}\psi_5^*(x)\right)\left(\frac{1}{2}p_{-1}\psi_{-1}(x) + \frac{1}{2}p_1\psi_1(x) + \frac{\sqrt{2}}{2}p_5\psi_5(x)\right)dx \\ &= \frac{1}{4}p_{-1} + \frac{1}{4}p_1 + \frac{1}{2}p_5 = \frac{1}{2}p_5 = \frac{1}{2}\hbar\frac{2\pi \cdot 5}{a} = \frac{5\hbar\pi}{a}. \end{aligned}$$

$$\begin{aligned} \langle p^2 \rangle &= \int \psi^*(x)\frac{\hbar}{i}\frac{d}{dx}\psi(x)dx = \int \left(\frac{1}{2}\psi_{-1}^*(x) + \frac{1}{2}\psi_1^*(x) + \frac{\sqrt{2}}{2}\psi_5^*(x)\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\left(\frac{1}{2}\psi_{-1}(x) + \frac{1}{2}\psi_1(x) + \frac{\sqrt{2}}{2}\psi_5(x)\right)dx \\ &= \int \left(\frac{1}{2}\psi_{-1}^*(x) + \frac{1}{2}\psi_1^*(x) + \frac{\sqrt{2}}{2}\psi_5^*(x)\right)\left(\frac{1}{2}(p_{-1})^2\psi_{-1}(x) + \frac{1}{2}(p_1)^2\psi_1(x) + \frac{\sqrt{2}}{2}(p_5)^2\psi_5(x)\right)dx \\ &= \frac{1}{4}(p_{-1})^2 + \frac{1}{4}(p_1)^2 + \frac{1}{2}(p_5)^2 = \frac{52\hbar^2\pi^2}{a^2}. \end{aligned}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{3\sqrt{3}\pi\hbar}{a}$$

6. (20 pts.) The rest mass of the electron neutrino ν_e is nearly zero so that its energy- momentum relation can be written as $E = pc$ where c is the speed of light in free space. Suppose that the particle number density of ν_e in a space is n and the interaction between the electron neutrinos is so small that the neutrinos can be taken as a non-interacting Fermi gas.

(a) Find the Fermi wave number k_F of the neutrino gas.

(b) Find the Fermi energy E_F Fermi temperature T_F of the neutrino gas.

(c) Find the average energy per particle $\overline{E}$ of the electron neutrinos under zero temperature.

Solution: (a)

$$N = g_s \frac{\frac{4}{3}\pi k_F^2}{\left(\frac{2\pi}{L}\right)^3} = 2V \frac{\frac{4}{3}\pi k_F^2}{(2\pi)^3} = V \frac{k_F^3}{3\pi^2}.$$

$$k_F^3 = 3\pi^2 \frac{N}{V} = 3\pi^2 n, \quad k_F = \left(3\pi^2 n\right)^{\frac{1}{3}}$$

(b) (1) $$g_s \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} = 2V \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{8\pi V}{c^3 h^3} E^2 dE = g(E)dE$$

$$N = \int_0^\infty f_{FD}(E) g(E) dE = \int_0^\infty \theta(E - E_F) g(E) dE$$
$$= \int_0^{E_F} \frac{8\pi V}{c^3 h^3} E^2 dE = \frac{8\pi V}{c^3 h^3} \frac{1}{3} E_F^3$$

$$E_F^3 = c^3 h^3 \frac{3N}{8\pi V} = \frac{3}{8\pi} c^3 h^3 n, \quad E_F = \frac{ch}{2}\left(\frac{3n}{\pi}\right)^{\frac{1}{3}}, \quad T_F = \frac{E_F}{k_B} == \frac{ch}{2k_B}\left(\frac{3n}{\pi}\right)^{\frac{1}{3}}.$$

(2) $$E_F = c\hbar k_F = c\hbar\left(3\pi^2 n\right)^{\frac{1}{3}}, \quad T_F = \frac{E_F}{k_B} = \frac{c\hbar}{k_B}\left(3\pi^2 n\right)^{\frac{1}{3}}.$$

(c)

$$\overline{E} = \frac{\int_0^\infty E f_{FD}(E) g(E) dE}{\int_0^\infty f_{FD}(E) g(E) dE} = \frac{\int_0^\infty E\theta(E - E_F) g(E) dE}{\int_0^\infty \theta(E - E_F) g(E) dE}.$$
$$= \frac{\int_0^{E_F} E^3 dE}{\int_0^{E_F} E^2 dE} = \frac{3}{4} E_F$$