1	2	3	4	5	6	Sum
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Name___________________________ Jan 15, 2016 Friday

SID_____________________________ School/Department_____________________

University Physics

Final Examination

Kuang Yaming Honors School, Nanjing University

$$\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt,\ \Gamma(\nu+1)=\nu\Gamma(\nu),\ \Gamma(1)=1,\ \Gamma(1/2)=\sqrt{\pi}.$$

c	h	e	m_e	u	N_A	k_B
3.0×10^8 m/s	6.63×10^{-34} J·s 1.24×10^3eV ·nm/c	1.6×10^{-19} C	9.11×10^{-31} kg 0.511MeV/c^2	931.494MeV/c^2	6.02×10^{23} mol^{-1}	8.617×10^{-5}eV/K

The quark combinations of some particles: p—uud n—udd Λ^0 – uds Σ^+ – uus

Quantum numbers of three quarks:

Quark symbols	Charge numbers	Spin	Baryon numbers	Strangeness
u	2/3	1/2	1/3	0
d	−1/3	1/2	1/3	0
s	−1/3	1/2	1/3	−1

Select five out of the following six problems.

1.(20 pts.) When neutrons are in thermal equilibrium within a reactor, they satisfy the equipartition theorem.

(a) If the temperature of the reactor is 300 K, what is the average kinetic energy $\bar{\varepsilon}$ of such a thermal neutron?

(b) The mass of neutron is known to be 1.0087 u, What is the corresponding de Broglie wavelength λ?

Solution: (a) According to the equipartition theorem (or the law of equipartition) of classical statistical mechanics, the average kinetic energy per degree of freedom is $1/2\,k_BT$. Here, the translational motion of the neutrons is in the 3 dimensional space and the degree of freedom is 3. Thus the average kinetic energy is

$$\bar{\varepsilon}=\frac{3}{2}k_BT=\frac{3}{2}\times8.617\times10^{-5}\,\text{eV/K}\times300\text{K}=3.88\times10^{-2}\,\text{eV}.\quad (\bar{\varepsilon}=\frac{3}{2}k_BT=\frac{3}{2}\cdot\frac{1}{40}\text{eV}=\frac{3}{80}\text{eV})$$

(b) Since $\bar{\varepsilon}<<m_nc^2$, it is the non-relativistic case. Thus the corresponding average momentum of the neutrons is

$\bar{p}=\sqrt{2m_n\bar{\varepsilon}}$ and

$$\lambda=\frac{h}{\bar{p}}=\frac{hc}{c\sqrt{2m_n\bar{\varepsilon}}}=\frac{hc}{\sqrt{2m_nc^2\bar{\varepsilon}}}=\frac{1240\text{nm}\cdot\text{eV}}{\sqrt{2\cdot1.0087\cdot931.5\text{eV}\cdot3.88\times10^{-2}\,\text{eV}}}=0.145\text{nm}.$$

(For the relativistic case, using the relations $E=mc^2=m_0c^2+E_K$ (mass energy=rest (mass) energy+kinetic energy) and $E^2=m_0^2c^2+p^2c^2$, we obtain

$$p=\frac{1}{c}\sqrt{E^2-m_0^2c^4}=\frac{1}{c}\sqrt{(E-m_0c^2)(E-m_0c^2)}=\frac{1}{c}\sqrt{E_K(E_K+2m_0c^2)}$$

and
$$\lambda=\frac{h}{p}=\frac{hc}{\sqrt{E_K(E_K+2m_0c^2)}}.$$

For the non-relativistic case, $E_K<<m_0c^2$, we have

$$\lambda = \frac{hc}{\sqrt{E_K\left(E_K + 2m_0c^2\right)}} = \frac{hc}{\sqrt{2m_0c^2E_K\left(1+\frac{E_K}{2m_0c^2}\right)}} \approx \frac{h}{\sqrt{2mE_K}}.$$

<u>For the extreme-relativistic case,</u> $E = pc$, we have $\lambda = \frac{hc}{E}$.

2. (20 pts.) After long effort, in 1902, Marie and Pierre Curie succeeded in separating from uranium ore the first substantial quantity of radium, one decigram of pure $RaCl_2$(radium chloride). Actually the radium they obtained was the radioactive isotope ^{226}Ra, which has a half-life of 1600 a.

(a) How many nuclei of ^{226}Ra had they isolated?

(b) What is the average life time τ of the isotope ^{226}Ra.

(c) What is the decay rate R of their sample at a late time t?

(The atomic mass of Cl(chlorine) is 35.5u .)

Solution: (a) The number of isolated ^{226}Ra nuclei is

$$N_0 = \frac{m}{M(\mathrm{RaCl_2})}N_A = \frac{0.1g}{(226+2\times35.5)\mathrm{g/mol}}\times 6.02\times10^{23}\,\mathrm{mol^{-1}} = 2.0\times10^{20}.$$

(b) $N = N_0e^{-\lambda t}$, $\lambda = \frac{\ln 2}{t_{1/2}}$, $-dN = \lambda N_0 e^{-\lambda t}dt$

$$\tau = \frac{\int t(-dN)}{N_0} = \int_0^\infty \lambda t e^{-\lambda t}dt \overset{u=\lambda t}{=} \frac{1}{\lambda}\int_0^\infty ue^{-u}du = -\frac{1}{\lambda}\int_0^\infty ud(e^{-u}) = -\frac{1}{\lambda}ue^{-u}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-u}du$$
$$= -\frac{1}{\lambda}e^{-u}\Big|_0^\infty = \frac{1}{\lambda} = \frac{t_{1/2}}{\ln 2} = 2.04\times10^3\,\mathrm{a}.$$

(c) $R(t) = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t} = R_0 e^{-\lambda t}$, $\lambda = \frac{\ln 2}{t_{1/2}} = 0.433\times10^{-3}\,\mathrm{a^{-1}} = 1.38\times10^{-11}\,\mathrm{s^{-1}}$,

$R_0 = 2.76\times10^9\,\mathrm{s^{-1}}$ (Bq Becquerel)

3.(20 pts.) An electron of a stationary hydrogen atom passes from the fifth level to the ground state, with a photon emitted in the process.

(a) Calculate the energy E_1 of the ground state and the energy E_5 of the fifth level, according to the Bohr theory.

(b) Considering the recoil of the hydrogen atom, what is the energy E_γ of the emitted photon, to the order of $(E_5 - E_1)^2/mc^2$ where $m = 938.782\,\mathrm{MeV}/c^2$ is the rest mass of the hydrogen atom?

(c) To the same order of magnitude, what is the recoil energy E_{recoil} of the hydrogen atom?

Solution: (a) $E_n = -13.6\frac{1}{n^2}\mathrm{eV}$, $E_1 = -13.6\mathrm{eV}$, $E_5 = -0.544\mathrm{eV}$.

(b) The conservation of momentum gives

$$p_\gamma = p_H. \tag{1}$$

The conservation of energy gives

$$\sqrt{m^2c^4 + p_H^2c^2} + E_\gamma = mc^2 + \left(E_5 - E_1\right), \quad E_\gamma = p_\gamma c.$$

Substituting $p_H = p_\gamma$ into above equation, we have

$$\sqrt{m^2c^4+E_\gamma^2}+E_\gamma=mc^2+(E_5-E_1) \quad \text{or} \quad \sqrt{m^2c^4+E_\gamma^2}=mc^2+(E_5-E_1)-E_\gamma .$$

To square the both sides of above equation, we have

$$m^2c^4+E_\gamma^2=m^2c^4+(E_5-E_1)^2+E_\gamma^2+2mc^2(E_5-E_1)-2mc^2E_\gamma-2(E_5-E_1)E_\gamma$$

or

$$\begin{aligned}2[mc^2+(E_5-E_1)]E_\gamma&=(E_5-E_1)^2+2mc^2(E_5-E_1)\\&=2[mc^2+(E_5-E_1)](E_5-E_1)-(E_5-E_1)^2\end{aligned}.$$

Thus

$$E_\gamma=E_5-E_1-\frac{(E_5-E_1)^2}{2[mc^2+(E_5-E_1)]}\approx E_5-E_1-\frac{(E_5-E_1)^2}{2mc^2} \qquad (\text{Noting that } E_5-E_1<<mc^2)$$

(c)

$$\begin{aligned}E_{\text{recoil}}&=\sqrt{m^2c^4+p_H^2c^2}-mc^2=\sqrt{m^2c^2+p_\gamma^2c^2}-mc^2=mc^2\sqrt{1+\frac{p_\gamma^2}{m^2c^2}}-mc^2\\&\approx\frac{p_\gamma^2}{2m}=\frac{E_\gamma}{2mc^2}\approx\frac{(E_5-E_1)^2}{2mc^2}.\end{aligned}$$

4.(20 pts.) Black body radiation can be treated as an ideal gas of photons. As such, its density of states $D(\nu)$ in the frequency ν space is $D(\nu)=V8\pi\nu^2c^{-3}$ and the number of quantum states within the interval of frequency ν to $\nu+d\nu$ is $V8\pi\nu^2c^{-3}d\nu$, where V is the volume of the photon gas.

(a) Show that the internal energy of the photon gas is $U=\alpha T^4V$ where T is temperature of the system and $\alpha=k_B^4\Gamma(4)\varsigma(4)/\pi^2c^3\hbar^3$.

(b) Find the heat capacity at constant volume C_V of the system.

(c) Find the Helmholtz free energy F and the Gibbs free energy G of the system.

(The fundamental equation of thermodynamics here can be written as $dU=TdS-pdV$.

$$\int_0^{+\infty}\frac{x^{\nu-1}}{e^x-1}dx=\Gamma(\nu)\varsigma(\nu), \nu>0.\ )$$

Solution: (a)

$$\begin{aligned}U&=\int_0^\infty\frac{h\nu}{e^{\beta h\nu}-1}D(\nu)d\nu=\frac{8\pi hV}{c^3}\int_0^\infty\frac{\nu^3}{e^{\beta h\nu}-1}d\nu\overset{\beta h\nu=x}{=}\frac{8\pi hV}{c^3}\frac{1}{(\beta h)^4}\int_0^\infty\frac{x^3}{e^x-1}dx\\&=\frac{8\pi Vk_B^4T^4}{c^3h^3}\int_0^\infty\frac{x^3}{e^x-1}dx=\frac{8\pi Vk_B^4T^4}{c^3h^3}\Gamma(4)\zeta(4)=\alpha T^4V.\end{aligned}$$

(b) $$C_V=\left(\frac{\partial U}{\partial T}\right)_V=4\alpha T^3V .$$

(c) Using the relation $C_V=T\left(\frac{\partial S}{\partial T}\right)_V=4\alpha T^3V$, we have

$$S=\int_0^T\frac{C_V}{T}dT=S=\int_0^T\frac{4\alpha T^3V}{T}dT=\frac{4}{3}\alpha T^3V .$$

Therefore, the Helmholtz free energy is

$$F=U-TS=\alpha T^4V-\frac{4}{3}\alpha T^4V=-\frac{1}{3}\alpha T^4V .$$

Using the relation $dF = -SdT - pdV$, we have

$$p = -\left(\frac{\partial F}{\partial V}\right)_T = -\frac{1}{3}\alpha T^3$$

Therefore, the Gibbs free energy is

$$G = U - TS + pV = F + pV = 0.$$

(Or $G = N\mu = 0$, where μ is the chemical potential of the photon gas (per particle) and is zero.)

5. (20 pts.) A one-dimensional simple harmonic oscillator with mass m and natural angular frequency ω is found to stay in the quantum state of following normalized wave function

$$\psi = c\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}\alpha^2 x^2} + \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \alpha x e^{-\frac{1}{2}\alpha^2 x^2},$$

where $\alpha = (m\omega/\hbar)^{1/2}$, and c is a positive constant to be determined.

(a) Determine the constant c.

(b) Is the ψ an eigenfunction of the Hamiltonian of the oscillator?

(c) Find $\langle E \rangle$, the average energy of the oscillator.

(d) Find ΔE, the uncertainty or fluctuation of the energy of the oscillator.

Solution: (a) From

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} \left[c\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \mathrm{e}^{-\frac{1}{2}\alpha^2 x^2} + \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \alpha x \mathrm{e}^{-\frac{1}{2}\alpha^2 x^2} \right]^2 dx \\
&= \int_{-\infty}^{\infty} \left(c^2 \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2} + 2c\frac{\alpha^2 x}{\sqrt{\pi}} e^{-\alpha^2 x^2} + \frac{\alpha^3 x^2}{\sqrt{\pi}} e^{-\alpha^2 x^2} \right) dx \\
&= \int_{-\infty}^{\infty} \left(c^2 \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2} + \frac{\alpha^3 x^2}{\sqrt{\pi}} e^{-\alpha^2 x^2} \right) dx \overset{\alpha x = u}{=} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(c^2 e^{-u^2} + u^2 e^{-u^2} \right) du \\
&= \frac{c^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du \overset{u^2 = t}{=} \frac{c^2}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} dt + \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt \\
&= \frac{c^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) + \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = c^2 + \frac{1}{2},
\end{aligned}$$

we obtain $c^2 = \dfrac{1}{2}$ or $c = \dfrac{\sqrt{2}}{2}$.

(b) $\hat{H} = -\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2} + \dfrac{1}{2}m\omega^2 x^2$

Noting that $\dfrac{d^2}{dx^2}\left(\mathrm{e}^{-\frac{1}{2}\alpha^2 x^2}\right) = \dfrac{d}{dx}\left(-\alpha^2 x \mathrm{e}^{-\frac{1}{2}\alpha^2 x^2}\right) = -\alpha^2 \mathrm{e}^{-\frac{1}{2}\alpha^2 x^2} + \alpha^4 x^2 \mathrm{e}^{-\frac{1}{2}\alpha^2 x^2}$,

and

$$\begin{aligned}
\frac{d^2}{dx^2}\left(x e^{-\frac{1}{2}\alpha^2 x^2}\right) &= \frac{d}{dx}\left(e^{-\frac{1}{2}\alpha^2 x^2} - \alpha^2 x^2 e^{-\frac{1}{2}\alpha^2 x^2}\right) = -\alpha^2 x e^{-\frac{1}{2}\alpha^2 x^2} - 2\alpha^2 x e^{-\frac{1}{2}\alpha^2 x^2} + \alpha^4 x^3 e^{-\frac{1}{2}\alpha^2 x^2}, \\
&= -3\alpha^2 x e^{-\frac{1}{2}\alpha^2 x^2} + \alpha^4 x^3 e^{-\frac{1}{2}\alpha^2 x^2}
\end{aligned}$$

we have

$$\hat{H}\mathrm{e}^{-\frac{1}{2}\alpha^2x^2} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(e^{-\frac{1}{2}\alpha^2x^2}\right)+\frac{1}{2}m\omega^2x^2\left(e^{-\frac{1}{2}\alpha^2x^2}\right)=\frac{\alpha^2\hbar^2}{2m}e^{-\frac{1}{2}\alpha^2x^2}-\frac{\hbar^2\alpha^4}{2m}x^2e^{-\frac{1}{2}\alpha^2x^2}+\frac{1}{2}m\omega^2x^2e^{-\frac{1}{2}\alpha^2x^2}$$

$$=\frac{\frac{m\omega}{\hbar}\hbar^2}{2m}e^{-\frac{1}{2}\alpha^2x^2}-\frac{\hbar^2\left(\frac{m\omega}{\hbar}\right)^2}{2m}x^2e^{-\frac{1}{2}\alpha^2x^2}+\frac{1}{2}m\omega^2x^2e^{-\frac{1}{2}\alpha^2x^2}=\frac{\hbar\omega}{2}e^{-\frac{1}{2}\alpha^2x^2};$$

$$\hat{H}xe^{-\frac{1}{2}\alpha^2x^2} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(xe^{-\frac{1}{2}\alpha^2x^2}\right)+\frac{1}{2}m\omega^2x^2\left(xe^{-\frac{1}{2}\alpha^2x^2}\right)$$

$$=\frac{3\alpha^2\hbar^2}{2m}xe^{-\frac{1}{2}\alpha^2x^2}-\frac{\hbar^2\alpha^4}{2m}x^3e^{-\frac{1}{2}\alpha^2x^2}+\frac{1}{2}m\omega^2x^3e^{-\frac{1}{2}\alpha^2x^2}$$

$$=\frac{3\frac{m\omega}{\hbar}\hbar^2}{2m}xe^{-\frac{1}{2}\alpha^2x^2}-\frac{\hbar^2\left(\frac{m\omega}{\hbar}\right)^2}{2m}x^3e^{-\frac{1}{2}\alpha^2x^2}+\frac{1}{2}m\omega^2x^3e^{-\frac{1}{2}\alpha^2x^2}=\frac{3\hbar\omega}{2}xe^{-\frac{1}{2}\alpha^2x^2}.$$

Therefore

$$\hat{H}\psi=\frac{\hbar\omega}{2}\left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2}\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}+\frac{3\hbar\omega}{2}\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\alpha x\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}\neq E\psi,$$

ψ is not an eigenfunction of $\hat{H}$.

(c)

$$\langle E\rangle=\int_{-\infty}^{\infty}\psi^*\hat{H}\psi dx$$
$$=\int_{-\infty}^{\infty}\left[\left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2}e^{-\frac{1}{2}\alpha^2x^2}+\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\alpha xe^{-\frac{1}{2}\alpha^2x^2}\right]\left[\frac{\hbar\omega}{2}\left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2}\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}+\frac{3\hbar\omega}{2}\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\alpha x\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}\right]dx$$
$$=\frac{\hbar\omega}{2}\int_{-\infty}^{\infty}\frac{\alpha}{2\sqrt{\pi}}e^{-\alpha^2x^2}dx+2\hbar\omega\int_{-\infty}^{\infty}\frac{\alpha^2x}{\sqrt{2\pi}}e^{-\alpha^2x^2}dx+\frac{3\hbar\omega}{2}\int_{-\infty}^{\infty}\frac{\alpha^3x^2}{\sqrt{\pi}}e^{-\alpha^2x^2}dx=\frac{\hbar\omega}{2}\cdot\frac{1}{2}+\frac{3\hbar\omega}{2}\cdot\frac{1}{2}=\hbar\omega.$$

(d)

$$\hat{H}^2\psi=\hat{H}\left(\hat{H}\psi\right)=\left(\frac{\hbar\omega}{2}\right)^2\left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2}\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}+\left(\frac{3\hbar\omega}{2}\right)^2\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\alpha x\mathrm{e}^{-\frac{1}{2}\alpha^2x^2},$$

$$\langle E^2\rangle=\int_{-\infty}^{\infty}\psi^*\hat{H}^2\psi dx$$
$$=\int_{-\infty}^{\infty}\left[\left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2}e^{-\frac{1}{2}\alpha^2x^2}+\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\alpha xe^{-\frac{1}{2}\alpha^2x^2}\right]\left[\left(\frac{\hbar\omega}{2}\right)^2\left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2}\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}+\left(\frac{3\hbar\omega}{2}\right)^2\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\alpha x\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}\right]dx$$
$$=\left(\frac{\hbar\omega}{2}\right)^2\int_{-\infty}^{\infty}\frac{\alpha}{2\sqrt{\pi}}e^{-\alpha^2x^2}dx+\left(\frac{3\hbar\omega}{2}\right)^2\int_{-\infty}^{\infty}\frac{\alpha^3x^2}{\sqrt{\pi}}e^{-\alpha^2x^2}dx=\frac{5}{4}(\hbar\omega)^2.$$

$$\Delta E=\sqrt{\langle E^2\rangle-\langle E\rangle^2}=\frac{\hbar\omega}{2}.$$

6. (20 pts.) For each of the following reactions state whether or not it is forbidden. If it is forbidden, state a conservation law that is violated.

(a) $\Lambda^0 \to p + \pi^0$

(b) $\overline{p} + p \to \mu^+ + e^-$

(c) $n \to p + e^- + \nu_e$

(d) $n + \nu_e \to p + e^-$

(e) $\Sigma^+ \to \Lambda^0 + \pi^+$

Answer:

(a)	$\Lambda^0 \to p + \pi^0$	Q	
(b)	$\overline{p} + p \to \mu^+ + e^-$	L_e and L_μ	
(c)	$n \to p + e^- + \nu_e$	L_e	
(d)	$n + \nu_e \to p + e^-$	not forbidden	weak interaction (neutrino)
(e)	$\Sigma^+ \to \Lambda^0 + \pi^+$	not forbidden	strong interaction

$uus \to uds + u\overline{d}$ (creation of $d\ \overline{d}$ pair)