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Jan. 11 2014, Saturday

Name____________________________

Department________________________ SID________________________

University Physics

Final Examination

Kuang Yaming Honors School, Nanjing University

Some physical constants: $h = 6.63\times10^{-34}\,\mathrm{J\cdot s}$, $1\mathrm{eV} = 1.60\times10^{-19}\,\mathrm{J}$, $m_e = 9.11\times10^{-31}\mathrm{kg} = 0.511\mathrm{MeV/c^2}$, $1\,\mathrm{u} = 1.660\times10^{-27}\mathrm{kg} = 931.5\mathrm{MeV/c^2}$, $hc = 1.24\times10^{3}\,\mathrm{eV\cdot nm}$.

Select five out of the following six problems.

1. A π^0 meson at rest decays into two photons ($\pi^0 \to \gamma + \gamma$).

(a) What type of interaction is responsible for the decay?

(b) What are the energy and momentum of each photon?

(c) What is the wavelength λ of the photons? (The mass of the π^0 is 135 MeV/c^2.)

Solution: (a) electromagnetic interaction

(b) $E_{\gamma_1} + E_{\gamma_2} = m_{\pi_0}c^2$, $\boldsymbol{p}_{\gamma_1} + \boldsymbol{p}_{\gamma_2} = 0 \;\Rightarrow\; E_{\gamma_1} = E_{\gamma_2} = \frac{1}{2}m_{\pi_0}c^2 = 67.5\,\mathrm{MeV}$

$$p_\gamma = \frac{E_\gamma}{c} = \frac{67.5\times10^6\times1.60\times10^{-19}\,\mathrm{J}}{3.0\times10^8\,\mathrm{ms^{-1}}} = 3.60\times10^{-20}\mathrm{kgms^{-1}}.$$

(c) $\lambda = \dfrac{h}{p_\gamma} = 1.84\times10^{-14}\,\mathrm{m}$.

2. Consider a one-dimensional simple harmonic oscillator of mass m and natural circular frequency ω.

(a) Write down the Hamiltonian $\hat{H}$ of the oscillator.

(b) Show that the wave function $\psi(x) = Nx\exp\left(-\dfrac{\alpha^2x^2}{2}\right)$, where $\alpha = \sqrt{\dfrac{m\omega}{\hbar}}$ and N is a constant, is an eigenfunction of the Hamiltonian $\hat{H}$ of the oscillator.

(c) Find $\langle x\rangle$, the mean value of the position of the particle, if the particle is in the state described by the wave function in (b).

Solution: (a) $\hat{H} = -\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2} + \dfrac{1}{2}m\omega^2x^2$.

(b)

$$\hat{H}\psi(x) = N\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(xe^{-\frac{\alpha^2x^2}{2}}\right) + \frac{m\omega^2x^2}{2}\left(xe^{-\frac{\alpha^2x^2}{2}}\right)\right]$$

$$= N\left[-\frac{\hbar^2}{2m}\frac{d}{dx}\left(e^{-\frac{\alpha^2x^2}{2}} - \alpha^2x^2e^{-\frac{\alpha^2x^2}{2}}\right) + \frac{m\omega^2x^3}{2}e^{-\frac{\alpha^2x^2}{2}}\right]$$

$$= N\left[-\frac{\hbar^2}{2m}\left(-\alpha^2xe^{-\frac{\alpha^2x^2}{2}} - 2\alpha^2xe^{-\frac{\alpha^2x^2}{2}} + \alpha^4x^3e^{-\frac{\alpha^2x^2}{2}}\right) + \frac{m\omega^2x^3}{2}e^{-\frac{\alpha^2x^2}{2}}\right]$$

$$= N\left[\frac{3\hbar^2\alpha^2}{2m}xe^{-\frac{\alpha^2x^2}{2}} - \frac{\hbar^2\alpha^4}{2m}x^3e^{-\frac{\alpha^2x^2}{2}} + \frac{m\omega^2x^3}{2}e^{-\frac{\alpha^2x^2}{2}}\right]$$

$$= N\left[\frac{3\hbar^2\frac{m\omega}{\hbar}}{2m}xe^{-\frac{\alpha^2x^2}{2}} - \frac{\hbar^2\left(\frac{m\omega}{\hbar}\right)^2}{2m}x^3e^{-\frac{\alpha^2x^2}{2}} + \frac{m\omega^2x^3}{2}e^{-\frac{\alpha^2x^2}{2}}\right]$$

$$= N\left(\frac{3}{2}\hbar\omega xe^{-\frac{\alpha^2x^2}{2}}\right) = \frac{3}{2}\hbar\omega\psi(x) .$$

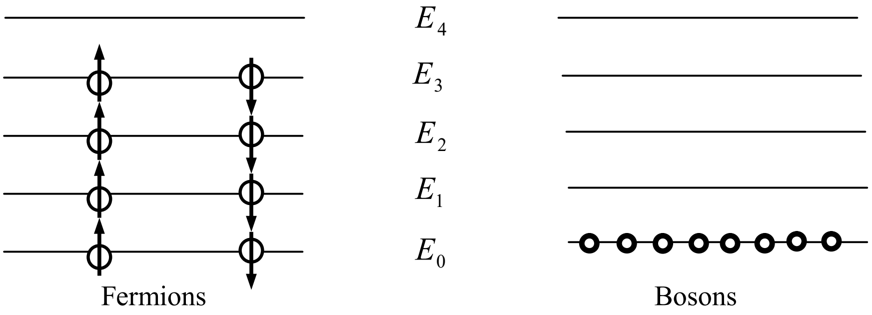
Therefore, $\psi(x)$ is an eigenfunction of $\hat{H}$.

(c) $\langle x\rangle = \dfrac{\int\psi^*(x)x\psi(x)dx}{\int\psi^*(x)\psi(x)dx} = 0.$

3. (a) The energy levels a particle in the one-dimensional harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$ can be given as $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0,1,2,\cdots$. If 8 identical fermions of spin 1/2 or 8 identical bosons of spin 0 are put in the harmonic potential, how the energy levels are filled in each case when the system is in the ground state? What is the ground state energy of the system in each case?

(b) The Fermi energy E_F for the electrons in copper is measured to be 7.00 eV. If the mass of the electron is $m = 0.511\,\mathrm{MeV/c^2}$, find the volume density n of the conducting electrons in copper.

Solution (a)



For fermions, we have

$$E_G = 2E_0 + 2E_1 + 2E_2 + 2E_3 = 16\hbar\omega.$$

For bosons, we have

$$E_G = 8E_0 = 4\hbar\omega.$$

(b) $E_F = \dfrac{\hbar^2 k_F^2}{2m} = \dfrac{\hbar^2}{2m}\left(3\pi^2 n\right)^{\frac{2}{3}}$, $n = \dfrac{1}{3\pi^2}\left(\dfrac{2mE_F}{\hbar^2}\right)^{\frac{3}{2}} = 8.4\times10^{28}\ \mathrm{m}^{-1}$.

4. (a) An unstable element of half-life $T_{1/2}$ is produced in a nuclear reactor at a constant rate R. What is the equilibrium quantity of the element? How much time is required to produce 50% of the equilibrium quantity of the element?

(b) In terms of the rest mass of the parent atom M_P and the rest mass of the daughter atom M_D, determine the Q-values for β^- decay, β^+ decay and electron capture (K-shell capture):

$$ {}^A_Z\mathrm{P} \rightarrow {}^{\ A}_{Z+1}\mathrm{D} + e^- + \bar{\nu}_e \quad (\beta^- \text{ decay}) $$

$$ {}^A_Z\mathrm{P} \rightarrow {}^{\ A}_{Z-1}\mathrm{D} + e^+ + \nu_e \quad (\beta^+ \text{ decay}) $$

$$ {}^A_Z\mathrm{P} + e^- \rightarrow {}^{\ A}_{Z-1}\mathrm{D} + \nu_e \quad (\text{electron capture}). $$

Solution: (a) The differential equation for the quantity of the element is

$$ dN/dt = R - \lambda N . $$

The equation can be rewritten as

$$ \frac{d(R-\lambda N)}{R-\lambda N} = -\lambda dt . $$

Integrating both sides of above equation we have

$$ \ln(R-\lambda N) = -\lambda t + C' $$

or $R - \lambda N = Ce^{-\lambda t}$. Substituting $N(t=0)=0$ into the equation, we obtain $C = R$. Then

$$ N = \frac{R}{\lambda}(1-e^{-\lambda t}) . $$

When $t \to +\infty$, $N \to R/\lambda$ and $dN/dt \to 0$. Thus, the equilibrium quantity is

$$ N_{eq} = \frac{R}{\lambda} = \frac{RT_{1/2}}{\ln 2} . $$

Setting N to $N_{eq}/2$, we have

$$ \frac{R}{\lambda}(1-e^{-\lambda t}) = \frac{1}{2}\frac{R}{\lambda} . $$

Then

$$ e^{-\lambda t} = \frac{1}{2}, $$

or

$$ \lambda t = \ln 2, \qquad t = \frac{\ln 2}{\lambda} = T_{1/2} . $$

(b)

$Q = (M_P - M_D)\,c^2$ (β^- decay)

$Q = (M_P - M_D - 2m_e)\,c^2$ (β^+ decay)

$Q = (M_P - M_D)\,c^2$ (electron capture).

5. The quark combinations of following particles are

$\pi^- - \mathrm{d}\bar{\mathrm{u}}$ $\pi^+ - \mathrm{u}\bar{\mathrm{d}}$ $\mathrm{K}^0 - \mathrm{d}\bar{\mathrm{s}}$ $\mathrm{K}^+ - \mathrm{u}\bar{\mathrm{s}}$ $\mathrm{K}^- - \mathrm{s}\bar{\mathrm{u}}$ $\mathrm{p} - \mathrm{uud}$ $\Xi^0 - \mathrm{uss}$ $\Omega^- - \mathrm{sss}$.

(a) Find the charge number Q, the baryon number B and the strangeness S of these particles.

(b) Determine if the following reactions are mainly mediated by the strong or weak interaction, or are forbidden. Where a reaction is forbidden, give the reason.

(1) $\mathrm{K}^- + \mathrm{p} \rightarrow \Omega^- + \mathrm{K}^+ + \mathrm{K}^0$

(2) $\Omega^- \rightarrow \Xi^0 + \pi^0$

(3) $\mathrm{K}^0 \rightarrow \pi^- + \mu^+ + \nu_\mu$

(4) $\mathrm{p} \rightarrow \mathrm{e}^+ + \gamma$

Quantum numbers of three quarks

Quark symbol	Charge number	Spin	Baryon number	Strangeness
u	$2/3$	$1/2$	$1/3$	0
d	$-1/3$	$1/2$	$1/3$	0
s	$-1/3$	$1/2$	$1/3$	-1

Solution: (a)

Particle	Q	B	S	*Particle*	Q	B	S
π^-	−1	0	0	π^+	1	0	0
K^0	0	0	1	K^+	1	0	1
K^-	−1	0	−1	p	+1	1	0
Ξ^0	0	1	−2	Ω^-	−1	1	−3

(b) (1) and (3) can occur. (2) and (4) are forbidden.

(1) is mainly mediated by the strong interaction and (3) is a weak decay.

(2) is forbidden since the charge number (or electric charge) is not conserved

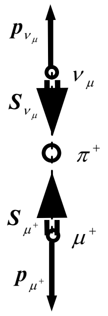
(4) is forbidden since both the baryon number and the electron lepton are not conserved.

6. Consider a spinless positive pion decaying at rest. (The initial angular momentum and linear momentum of the pion are zero.) A positive muon and a muon neutrino are emitted in opposite directions with equal and opposite momenta $\boldsymbol{p}_{\mu^+}$ and $\boldsymbol{p}_{\nu_\mu}$. Meanwhile, the two emitted particles have equal but opposite spins ($\boldsymbol{S}_{\mu^+}$ and $\boldsymbol{S}_{\nu_\mu}$) due to the angular momentum conservation.

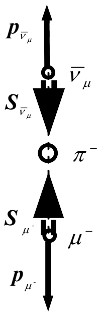
(a) Draw a figure to show the charge-conjugate process.

(b) Can the charge-conjugate process occur in nature? Why?

(c) What conclusion can you make if the charge-conjugate process can/cannot occur in nature?



Original process



Charge-conjugate process

Solution: (a) See the figure.

(b) The charge-conjugate process cannot occur in nature, since the anti-muon neutrino has the wrong helicity (chirality or handedness). (The helicity of a neutrino is always -1 (left-handed) while the helicity of a anti-neutrino is always +1 (right-handed.)

(c) The charge-conjugate process cannot occur in nature. Therefore, in the process, the charge conjugation is not conserved.