1	2	3	4	5	6	Sum
/20	/20	/20	/20	/20	/20	/100

Name ______________________ January 6th, 2008 Sunday

Department _________________

SID _______________________

University Physics

Final Examination

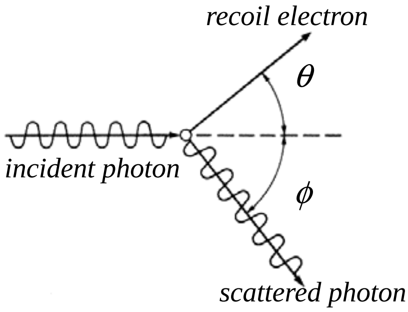
School of Intensive Instruction in Sciences and Arts, Nanjing University

Physical Constants

Electron charge magnitude	$e = 1.60\times10^{-19}$ C
Dielectric constant of vacuum	$\varepsilon_0 = 8.85\times10^{-12}$ F/m
Permeability of vacuum	$\mu_0 = 4\pi\times10^{-7}$ N/A^2
Vacuum speed of light	$c = 3.00\times10^{8}$ m/s
Electron rest mass	$m_e = 9.11\times10^{-31}$ kg $= 0.511$Mev/c^2
Proton rest mass	$m_p = 1.672\times10^{-27}$ kg $= 938.3$MeV/c^2
Neutron rest mass	$m_n = 1.675\times10^{-27}$ kg $= 939.6$MeV/c^2
Atomic mass unit	1 amu $= 1.660\times10^{-27}$ kg $= 931.5$MeV/c^2
Electron volt (eV)	1eV=1.60×10^{-19} J
Planck constant	$h = 6.63\times10^{-34}$ Js $\hbar = 1.06\times10^{-34}$ Js $= 0.66\times10^{-15}$ eVs
Boltzmann constant	$k_B = 1.38\times10^{-23}$ J/K $= 8.62\times10^{-5}$ eV/K
Bohr radius	$a_0 = 0.529$ Angstroms

Select five out of following six problems.

1. (20pts) In Compton scattering, an incoming X-ray photon of frequency ν scatters off an electron that is initially at rest. The direction of scattered photon makes an angle ϕ to the direction of the incident X-ray, and the electron gains energy and recoils in the direction that makes an angle θ to the direction of the incident photon as shown in the figure.



(a) Show that the increase of the wavelength of the scattered photon is given by

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\phi).$$

(b) Show that the scattering angle of the electron θ is given by

$$\cot\theta = \left(1 + \frac{h\nu}{m_e c^2}\right)\tan\frac{\phi}{2}.$$

Proof: The energy-momentum relations are

$$E_e^2 = c^2 p_e^2 + m_e^2 c^4, \quad E = cp, \quad E' = cp'. \tag{1}$$

The conservation of energy gives

$$(E - E' + m_e c^2)^2 = E_e^2 = c^2 p_e^2 + m_e^2 c^4. \tag{2}$$

The conservation of momentum gives

$$p_e^2 = |\boldsymbol{p} - \boldsymbol{p}'|^2 = p^2 - 2pp'\cos\phi + p'^2. \tag{3}$$

By substituting (1) and (3) into (2) , we have

$$(cp - cp' + m_e c^2)^2 = c^2(p^2 - 2pp'\cos\phi + p'^2) + m_e^2 c^4,$$

or

$$2(cp - cp')m_e c^2 = 2c^2 pp'(1-\cos\phi).$$

Dividing both sides with $2c^2 pp'$, we obtain

$$m_e c\left(\frac{1}{p'} - \frac{1}{p}\right) = 1 - \cos\phi \text{ .}$$

Noting that $p = h/\lambda$ and $p' = h/\lambda'$, we finally have

$$\lambda' - \lambda = \left(\frac{h}{m_e c}\right)(1-\cos\phi) \text{ .}$$

(b) From the result of (a), we have

$$\frac{\nu - \nu'}{\nu'} = \frac{h\nu}{m_e c^2}(1-\cos\phi)\text{ .}$$

The momentum conservation equation can be written in components as

$$\begin{aligned} \frac{h\nu}{c} &= \frac{h\nu'}{c}\cos\phi + p_e\cos\theta \\ \frac{h\nu'}{c}\sin\phi &= p_e\sin\theta \end{aligned} \text{ .}$$

Thus, we have

$$\begin{aligned} \cot\theta &= \frac{\nu - \nu'\cos\phi}{\nu'\sin\phi} = \frac{\nu - \nu'}{\nu'\sin\phi} + \frac{1-\cos\phi}{\sin\phi} \\ &= \left(1 + \frac{h\nu}{m_e c^2}\right)\frac{(1-\cos\phi)}{\sin\phi} = \left(1 + \frac{h\nu}{m_e c^2}\right)\frac{2\sin^2\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} \\ &= \left(1 + \frac{h\nu}{m_e c^2}\right)\tan\frac{\phi}{2}. \end{aligned}$$

2. (20 pts) An electron is moving in a one-dimensional infinite potential well

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ +\infty & \text{elsewhere} \end{cases}\text{ .}$$

(a) Write the Schrödinger's equation of the electron
(b) Obtain the energy levels and the corresponding stationary wave functions of the electron.
(c) Show that the expectation (average) value of the kinetic energy of a particle in one dimension can be expressed as

$$\left\langle \hat{T} \right\rangle \equiv \left\langle \frac{\hat{p}^2}{2m} \right\rangle = \frac{\hbar^2}{2m}\int \left|\frac{d\psi}{dx}\right|^2 dx$$

Solution: (a) The Schrödinger's equation can be written as

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \qquad (0 < x < a) \\ \psi &= 0 \qquad\qquad \text{elsewhere .} \end{aligned}$$

(b) The equation can be rewritten as

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad \left(k^2 \equiv \frac{2mE}{\hbar^2}\right)\text{.}$$

The general solution is

$$\psi(x) = A\sin(kx) + B\cos(kx) .$$

With the use of the boundary condition $\psi(0) = \psi(a) = 0$, the solution is

$$\psi_n(x) = A\sin k_n x , \quad k_n = \frac{n\pi}{a} , \ n = 1, \ 2, \ 3, \ \cdots .$$

The energy levels are

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 .$$

(c) The expectation value of the kinetic energy is

$$\left\langle \hat{T} \right\rangle = \int_{-\infty}^{+\nabla} \psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi(x)dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \psi^*(x)\frac{d^2\psi(x)}{dx^2}dx$$

$$= -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \psi^*(x)d\left(\frac{d\psi(x)}{dx}\right) = \left[-\frac{\hbar^2}{2m}\psi^*(x)d\left(\frac{d\psi(x)}{dx}\right)\right]_{-\infty}^{+\infty} + \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} d\left(\psi^*(x)\right)\frac{d\psi(x)}{dx}$$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \frac{d\psi^*(x)}{dx}\frac{d\psi(x)}{dx}dx = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left|\frac{d\psi(x)}{dx}\right|^2 dx$$

3. (20pts) The energy levels and the corresponding stationary wave functions of a particle in the one-dimensional harmonic potential $V(x) = 1/2\, kx^2$ can be given as

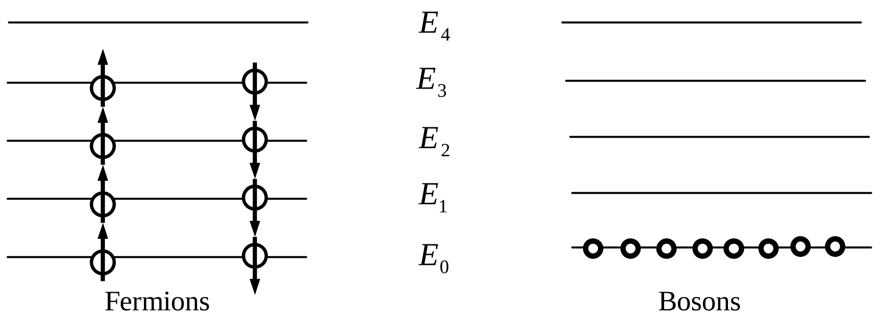
$$E_n = (n + 1/2)\hbar\omega , \quad \psi_n(x) = N_n e^{-\frac{\alpha^2 x^2}{2}} H_n(\alpha x), \quad n = 0,1,2,\cdots .$$

We put 8 identical fermions of spin 1/2 or 8 identical bosons of spin 0 in the harmonic potential.

(a) Draw a schematic diagram to show how the energy levels are filled in each case when the system is in the ground state.

(b) Find the ground state energy of the system in each case.

Solution (a)



(b) For fermions, we have

$$E_G = 2E_0 + 2E_1 + 2E_2 + 2E_3 = 16\hbar\omega .$$

For bosons, we have

$$E_G = 8E_0 = 4\hbar\omega .$$

4.(20pts) An unstable element of half-life $T_{1/2}$ is produced in a nuclear reactor at a constant rate R.

(a) What is the equilibrium quantity of the element?

(b) How much time is required to produce 50% of the equilibrium quantity of the element?

(c) In terms of the rest mass of the parent atom M_P and the rest mass of the daughter atom M_D , determine the Q-values for β^- decay, β^+ decay and electron capture (K-shell capture):

$${}^A_Z\mathrm{P} \rightarrow {}^{\ \ A}_{Z+1}\mathrm{D} + e^- + \bar{\nu}_e \quad (\beta^- \text{ decay})$$

$${}^A_Z\mathrm{P} \rightarrow {}^{\ \ A}_{Z-1}\mathrm{D} + e^+ + \nu_e \quad (\beta^+ \text{ decay})$$

$${}^A_Z\mathrm{P} + e^- \rightarrow {}^{\ \ A}_{Z-1}\mathrm{D} + \nu_e \quad (\text{electron capture}).$$

Solution: (a) The differential equation for the quantity of the element is

$$dN/dt = R - \lambda N \text{ .}$$

The equation can be rewritten as

$$\frac{d(R-\lambda N)}{R-\lambda N} = -\lambda dt \text{ .}$$

Integrating both sides of above equation we have

$$\ln(R-\lambda N) = -\lambda t + C'$$

or $R - \lambda N = Ce^{-\lambda t}$. Substituting $N(t=0)=0$ into the equation, we obtain $C = R$. Then

$$N = \frac{R}{\lambda}(1-e^{-\lambda t}) \text{ .}$$

When $t \to +\infty$, $N \to R/\lambda$ and $dN/dt \to 0$. Thus, the equilibrium quantity is

$$N_{eq} = \frac{R}{\lambda} = \frac{RT_{1/2}}{\ln 2} \text{ .}$$

(b) Setting N to $N_{eq}/2$, we have

$$\frac{R}{\lambda}(1-e^{-\lambda t}) = \frac{1}{2}\frac{R}{\lambda} \text{.}$$

Then

$$e^{-\lambda t} = \frac{1}{2} \text{,}$$

or

$$\lambda t = \ln 2 \text{,} \qquad t = \frac{\ln 2}{\lambda} = T_{1/2} \text{.}$$

(c)

$$Q = (M_P - M_D)\,c^2 \qquad (\beta^- \text{ decay})$$

$$Q = (M_P - M_D - 2m_e)\,c^2 \qquad (\beta^+ \text{ decay})$$

$$Q = (M_P - M_D)\,c^2 \qquad (\text{electron capture}).$$

5. The quark combinations of following particles are

$\pi^- - d\bar{u}$ $\quad \pi^+ - u\bar{d}$ $\quad K^0 - d\bar{s}$ $\quad K^- - s\bar{u}$ $\quad p - uud$ $\quad \Xi^0 - uss$ $\quad \Omega^- - sss$.

(a) Find the charge number Q, the baryon number B and the strangeness S of these particles.

(b) Determine if the following processes are mainly mediated by the strong or weak interaction, or are forbidden. Where a process is forbidden, give all of the reasons why this is the case.

(1) $K^- + p \to \Omega^- + K^+ + K^0$

(2) $\Omega^- \to \Xi^0 + \pi^0$

(3) $K^0 \to \pi^- + \mu^+ + \nu_\mu$

(4) $p \to e^+ + \gamma$

Quantum numbers of three quarks

Quark symbol	Charge number	Spin	Baryon number	Strangeness
u	$2/3$	$1/2$	$1/3$	0
d	$-1/3$	$1/2$	$1/3$	0
s	$-1/3$	$1/2$	$1/3$	-1

Solution: (a)

Particle	*Q*	*B*	*S*	*Particle*	*Q*	*B*	*S*
π^-	–1	0	0	p	+1	1	0
K^0	0	0	1	Ξ^0	0	1	–2
K^-	–1	0	–1	Ω^-	–1	1	–3

(b) (1) , (2) and (3) can occur. (4) is forbidden.

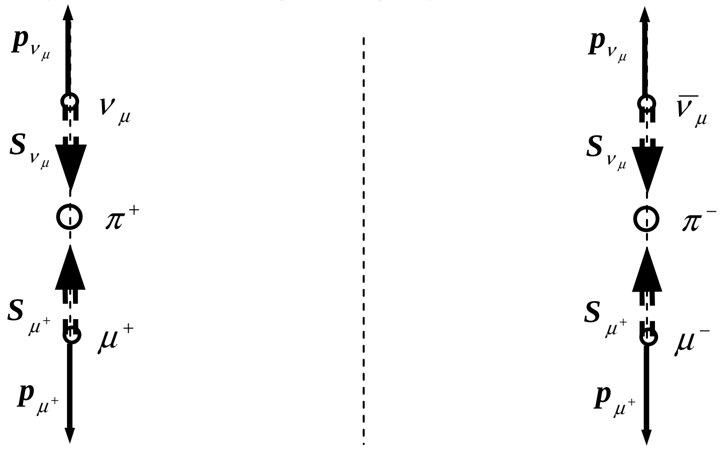
(2) is forbidden since both the baryon number and the electron lepton are not conserved.

6.(20pts) Consider a spinless positive pion decaying at rest. (The initial angular momentum and linear momentum of the pion are zero.) A positive muon and a muon neutrino are emitted in opposite directions with equal and opposite momenta. Meanwhile, the two emitted particles have equal but opposite spins due to the angular momentum conservation.

(a) Draw a figure to show the charge-conjugate process.

(b) Can the charge-conjugate process occur in nature? Why?

(c) What conclusion can you make if the charge-conjugate process can/cannot occur in nature?



Original process

Charge-conjugate process

Solution: (a) See the figure.

(b) The charge-conjugate process cannot occur in nature, since the anti-muon neutrino has the wrong helicity (handness).

(c) The charge-conjugate process cannot occur in nature, therefore, in the process, the charge conjugation is not conserved.