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November 10, 2019 Sunday

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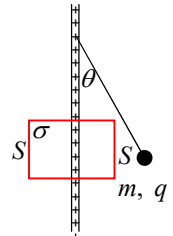
University Physics II
Midterm Examination
Kuang Yaming Honors School, Nanjing University

Select five out of following six problems.

1. (20pts.) An infinite non-conducting vertical planar sheet is charged homogeneously with areal electric charge density σ .

(a) Find the electric field \mathbf{E} everywhere, specifying the normal direction of the sheet as the direction of x -axis.

(b) A silk thread with one end suspended on the sheet and the other end attached to small charged sphere of mass m makes an angle $\theta = \pi/6$ with the sheet, as shown in the figure. What is the electric charge q on the sphere?



Solution: (a) Making a cylindrical Gaussian surface with two bases parallel to the sheet, as shown in the figure, and applying of Gauss's law gives

$$\epsilon_0(ES + ES) = \sigma S.$$

Thus, we obtain $E = \sigma/(2\epsilon_0)$, or

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\epsilon_0} \mathbf{e}_x, & x > 0, \\ -\frac{\sigma}{2\epsilon_0} \mathbf{e}_x, & x < 0. \end{cases}$$

(b) The equilibrium of the sphere requires

$$T \cos \theta = mg, \quad T \sin \theta = qE = \frac{q\sigma}{2\epsilon_0},$$

where T is the tension in the silk thread. By eliminating T in the above equations, we obtain

$$\tan \theta = \frac{q\sigma}{2mg\epsilon_0}.$$

Finally

$$q = \frac{2mg\epsilon_0}{\sigma} \tan \theta = \frac{2\sqrt{3}mg\epsilon_0}{3\sigma}.$$

2.(20pts.) An insulating rod of length l is positioned on the z -axis, from $z = -l/2$ to $z = l/2$, and it carries an electric charge of linear charge density λ .

(a) Calculate the electric potential $V = V(z)$ on the z -axis and within the region $z > l/2$ and $z < -l/2$. The potential at infinity is set to zero.

(b) Calculate the electric potential $V = V(x, y, z)$ everywhere, except on the z -axis. The potential at infinity is set to zero.

(c) Calculate the electric field $\mathbf{E} = \mathbf{E}(x, y, z)$ everywhere, except on the z -axis.

(Hint: $\int \frac{d\xi}{\sqrt{\xi^2 + 1}} = \ln|\xi + \sqrt{\xi^2 + 1}| + C$)

Solution: (a)

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{\lambda dz'}{|z - z'|}.$$

For $z > l/2$,

$$\begin{aligned} V(z) &= \frac{1}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{\lambda dz'}{z - z'} = -\frac{\lambda}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{dz'}{z' - z} = -\frac{\lambda}{4\pi\epsilon_0} \ln|z' - z| \Big|_{z'=-l/2}^{z'=l/2} \\ &= -\frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l/2 - z}{-l/2 - z} \right| = \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{z + l/2}{z - l/2} \right|. \end{aligned}$$

For $z < -l/2$

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{\lambda dz'}{z' - z} = \frac{\lambda}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{dz'}{z' - z} = \frac{\lambda}{4\pi\epsilon_0} \ln|z' - z| \Big|_{z'=-l/2}^{z'=l/2}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l/2 - z}{-l/2 - z} \right| = \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{z - l/2}{z + l/2} \right|.$$

(b)

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{\lambda dz'}{\sqrt{x^2 + y^2 + (z - z')^2}} = \frac{\lambda}{4\pi\epsilon_0 \sqrt{x^2 + y^2}} \int_{-l/2}^{l/2} \frac{dz'}{\sqrt{1 + \left(\frac{z' - z}{\sqrt{x^2 + y^2}}\right)^2}}.$$

Letting $\xi = \frac{z' - z}{\sqrt{x^2 + y^2}}$, we have

$$V(x, y, z) = \frac{\lambda}{4\pi\epsilon_0} \int_{\frac{-l/2 - z}{\sqrt{x^2 + y^2}}}^{\frac{l/2 - z}{\sqrt{x^2 + y^2}}} \frac{d\xi}{\sqrt{\xi^2 + 1}} = \frac{\lambda}{4\pi\epsilon_0} \ln \left| \xi + \sqrt{\xi^2 + 1} \right| \Big|_{\frac{-l/2 - z}{\sqrt{x^2 + y^2}}}^{\frac{l/2 - z}{\sqrt{x^2 + y^2}}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l/2 - z + \sqrt{(l/2 - z)^2 + x^2 + y^2}}{-l/2 - z + \sqrt{(l/2 + z)^2 + x^2 + y^2}} \right|.$$

(c) $\mathbf{E} = -\nabla V = -\mathbf{e}_x \frac{\partial V}{\partial x} - \mathbf{e}_y \frac{\partial V}{\partial y} - \mathbf{e}_z \frac{\partial V}{\partial z}$

$$\frac{\partial V}{\partial x} = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{l/2 - z + \sqrt{(l/2 - z)^2 + x^2 + y^2}} \frac{x}{\sqrt{(l/2 - z)^2 + x^2 + y^2}} - \frac{1}{-l/2 - z + \sqrt{(l/2 + z)^2 + x^2 + y^2}} \frac{x}{\sqrt{(l/2 + z)^2 + x^2 + y^2}} \right],$$

$$\frac{\partial V}{\partial y} = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{l/2 - z + \sqrt{(l/2 - z)^2 + x^2 + y^2}} \frac{y}{\sqrt{(l/2 - z)^2 + x^2 + y^2}} - \frac{1}{-l/2 - z + \sqrt{(l/2 + z)^2 + x^2 + y^2}} \frac{y}{\sqrt{(l/2 + z)^2 + x^2 + y^2}} \right],$$

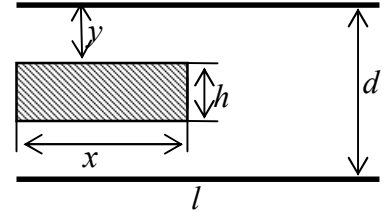
$$\frac{\partial V}{\partial z} = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{l/2 - z + \sqrt{(l/2 - z)^2 + x^2 + y^2}} \left(\frac{z - l/2}{\sqrt{(l/2 - z)^2 + x^2 + y^2}} - 1 \right) - \frac{1}{-l/2 - z + \sqrt{(l/2 + z)^2 + x^2 + y^2}} \left(\frac{z + l/2}{\sqrt{(l/2 + z)^2 + x^2 + y^2}} - 1 \right) \right].$$

3. (20pts.) As shown in the figure, a copper slab of thickness h is parallelly inserted into a square parallel-plate capacitor of separation d and side length l . The distance between the upper surface of the slab and the upper plate of the capacitor is y . The slab is rectangular of length l and width x ($x < l$).

(a) What is capacitance C of the system?

(b) If the charge on the capacitor is q , what is the electric potential energy U of the system?

(c) What is the horizontal force f_1 exerted on the slab by the electric field? And what is the vertical force f_2 ? Suppose the charge on the capacitor is fixed.



Solution:

(a) The system can be viewed as the parallel combination of two capacitors, the left part with the slab between two plates and the right part without. Thus

$$C = C_1 + C_2 = \frac{\epsilon_0 l x}{d - h} + \frac{\epsilon_0 l (l - x)}{d}$$

(b) When the capacitor is charged with q , the electric potential energy is

$$U = \frac{1}{2} \frac{q^2}{C} = \frac{q^2 d (d - h)}{2 \epsilon_0 l [(d - h)l + hx]}.$$

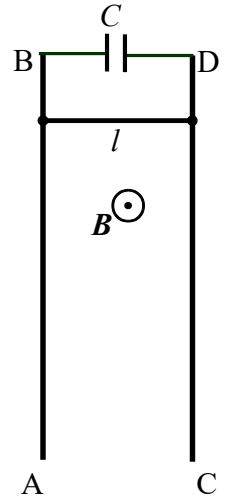
(c) The horizontal force acted on the slab can be calculated as

$$f_1 = - \left(\frac{\partial U}{\partial x} \right)_q = \frac{q^2 d (d - h) h}{2 \epsilon_0 l [(d - h)l + hx]^2},$$

and the vertical force is

$$f_2 = - \left(\frac{\partial U}{\partial y} \right)_q = 0.$$

4. (20pts.) A conducting rod of length l and mass m can slide without friction along two sufficiently long vertical parallel conducting rails AB and CD that are connected each other through a capacitor of capacitance C at the upper ends, as shown in the figure. Perpendicular to the plane of the rails, a uniform magnetic field of induction B is applied. Assume that the conducting rod and the conducting rails are made of ideal conductors of zero resistance and there is also no contact resistance. Initially, the rod is at rest and the capacitor is uncharged.



(a) Find the voltage V across the capacitor as a function of h , the lowered height of the rod from its initial position.

(b) If the capacitor is replaced by a resistor of resistance R , find the voltage V across the resistor as a function of time t .

(c) Discuss the meaning of the result (b) in the limit $t \rightarrow +\infty$.

Solution: (a) The conservation of energy gives

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}CV^2. \quad (1)$$

By applying Faraday's law, we have

$$V = -\frac{d\Phi_m}{dt} = -Blv, \quad \text{or} \quad v = -\frac{V}{Bl}.$$

Substituting above equation into (1), we have

$$mgh = \frac{1}{2} \left(\frac{m}{B^2 l^2} + C \right) V^2.$$

Thus

$$V = \sqrt{\frac{2mgh}{m + CB^2 l^2}} Bl.$$

(b) The equation of motion is

$$m \frac{dv}{dt} = mg - IBl$$

where $I = V/R$, $V = Blv$. Thus

$$\frac{m}{Bl} \frac{dV}{dt} = mg - \frac{Bl}{R} V \quad \text{or} \quad \frac{dV}{dt} = -\frac{B^2 l^2}{mR} \left(V - \frac{mgR}{Bl} \right).$$

By separating variables, we obtain

$$\frac{dV}{V - \frac{mgR}{Bl}} = -\frac{B^2 l^2}{mR} dt.$$

Integrating both sides of above equation, we have

$$\int_0^V \frac{dV}{V - \frac{mgR}{Bl}} = -\frac{B^2 l^2}{mR} \int_0^t dt \quad \text{or} \quad \ln \left(\frac{V - \frac{mgR}{Bl}}{-\frac{mgR}{Bl}} \right) = -\frac{B^2 l^2}{mR} t \quad \text{or} \quad V = \frac{mgR}{Bl} \left[1 - \exp \left(-\frac{B^2 l^2}{mR} t \right) \right].$$

(c) When $t \rightarrow \infty$, we have $V = \frac{mgR}{Bl}$. In this case, we have

$$P = mgv = \frac{mgV}{Bl} = \frac{V \cdot V}{R} = \frac{V^2}{R} = I^2 R.$$

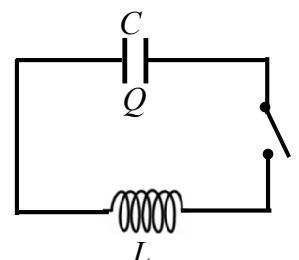
That is, the power of the gravity is just the power of Joule dissipation of the resistor.

5. (20pts.) In the circuit shown in the accompanying figure, the capacitor of capacitance C is initially charged with electric charge Q . The switch is closed at time $t = 0$ and the electric current begins to flow through the inductor of inductance L .

(a) Write down the equation of oscillation for the charge q on the capacitor. Assume the resistance in the circuit is negligible.

(b) Find the solution of charge q on the capacitor as a function of time t .

(c) Find the total electromagnetic energy U stored in the capacitor and inductor at time t .



Solution: (a) When the switch is closed, we have

$$\frac{q}{C} = -L \frac{dI}{dt}.$$

Substituting $I = \frac{dq}{dt}$ into above equation, we have $\frac{q}{C} = -L \frac{d^2q}{dt^2}$, or

$$\frac{d^2q}{dt^2} + \omega^2 q = 0, \quad \omega^2 = \frac{1}{LC}. \quad (1)$$

(b) The general solution of (1) can be written as

$$q(t) = A \cos \omega t + B \sin \omega t.$$

According to the problem, the initial conditions are $q(t=0) = Q$ and $I(t=0) = \left. \frac{dq}{dt} \right|_{t=0} = 0$. Substituting the initial conditions into the general solution, we obtain

$$q(t) = Q \cos \omega t = Q \cos \left(\frac{t}{\sqrt{LC}} \right).$$

(c) The electric current in the circuit is

$$I(t) = \frac{dq}{dt} = -\frac{Q}{\sqrt{LC}} \sin \left(\frac{t}{\sqrt{LC}} \right).$$

Thus, the total electromagnetic energy is

$$U = U_C + U_L = \frac{q^2}{2C} + \frac{1}{2} LI^2 = \frac{Q^2}{2C} \cos^2 \left(\frac{t}{\sqrt{LC}} \right) + \frac{1}{2} L \frac{Q^2}{LC} \sin^2 \left(\frac{t}{\sqrt{LC}} \right) = \frac{Q^2}{2C}.$$

6. (20pts.) An electromagnetic wave travels in free space (i.e., in the vacuum), and its magnetic induction \mathbf{B} takes the form

$$\mathbf{B} = (b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \cos(kz - \omega t)$$

where b_1 , b_2 , ω and k are all constants, $\omega = ck$ with c being the speed of light in the vacuum, \mathbf{e}_x and \mathbf{e}_y are the unit vectors along the x - and y - axis, respectively.

(a) Find, by applying Maxwell equations, the electric field \mathbf{E} and the displacement current density \mathbf{j}_D .

(b) What is the Poynting vector \mathbf{S} of this wave? What is the average Poynting vector $\langle \mathbf{S} \rangle$ over one period of time?

Solution: (a) Assume that $\mathbf{E} = \mathbf{E}_0 \cos(kz - \omega t)$, where \mathbf{E}_0 is to be determined. Ampere-Maxwell law is

$$\nabla \times \mathbf{H} = \mathbf{j}_{free} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}_{free} + \mathbf{j}_D.$$

In the vacuum, $\mathbf{j}_{free} = 0$, $\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$, $\mathbf{D} = \epsilon_0 \mathbf{E}$. Thus, we have

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \quad (1)$$

Calculating the rotation (or curl) of \mathbf{B} and the partial derivative of \mathbf{E} to t , we obtain

$$\begin{aligned} \nabla \times \mathbf{B} &= \nabla \times [(b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \cos(kz - \omega t)] = (b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \times \nabla [\cos(kz - \omega t)] \\ &= (b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \times \frac{\partial [\cos(kz - \omega t)]}{\partial z} \mathbf{e}_z = -k(b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \times \mathbf{e}_z \sin(kz - \omega t) \\ &= k(b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) \sin(kz - \omega t); \end{aligned}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \mathbf{E}_0 \frac{\partial \cos[(kz - \omega t)]}{\partial t} = \omega \mathbf{E}_0 \sin(kz - \omega t).$$

Substituting above expressions in to (1), we have

$$k(b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) = \frac{\omega}{c^2} \mathbf{E}_0 \quad \text{or} \quad \mathbf{E}_0 = \frac{c^2 k}{\omega} (b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) = c(b_2 \mathbf{e}_x - b_1 \mathbf{e}_y).$$

Finally $\mathbf{E} = c(b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) \cos(kz - \omega t)$ and $\mathbf{j}_D = \frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \omega c (b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) \sin(kz - \omega t)$.

[Hint: $\nabla \times (\mathbf{f}\varphi) = (\nabla \times \mathbf{f})\varphi + \mathbf{f} \times \nabla \varphi$.

The rotation can also be calculated as follows

$$\nabla \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 \cos(kz - \omega t) & b_2 \cos(kz - \omega t) & 0 \end{vmatrix} = k(b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) \sin(kz - \omega t).$$

(b) The Poynting vector is

$$\begin{aligned} \mathbf{S} &= \mathbf{E} \times \mathbf{H} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \\ &= \frac{1}{\mu_0} [c(b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) \cos(kz - \omega t)] \times [(b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \cos(kz - \omega t)] \\ &= \frac{c}{\mu_0} (b_2 \mathbf{e}_x - b_1 \mathbf{e}_y) \times (b_1 \mathbf{e}_x + b_2 \mathbf{e}_y) \cos^2(kz - \omega t) \\ &= \frac{c}{\mu_0} (b_1^2 + b_2^2) \cos^2(kz - \omega t) \mathbf{e}_z. \end{aligned}$$

The average Poynting vector $\langle \mathbf{S} \rangle$ over one period of time is

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{T} \int_0^T \mathbf{S} dt = \frac{c}{\mu_0} (b_1^2 + b_2^2) \mathbf{e}_z \left(\frac{1}{T} \int_0^T \cos^2(kz - \omega t) dt \right) \\ &= \frac{c}{\mu_0} (b_1^2 + b_2^2) \mathbf{e}_z \left\{ \frac{1}{T} \int_0^T \frac{1 + \cos[2(kz - \omega t)]}{2} dt \right\} = \frac{c}{2\mu_0} (b_1^2 + b_2^2) \mathbf{e}_z. \end{aligned}$$