1	2	3	4	5	6	Total
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Name: ______________________________ November 11, 2018 Sunday

Department/School:____________________ SID:__________________________________

University Physics II

Midterm Examination

Kuang Yaming Honors School, Nanjing University

<u>Select five out of following six problems.</u>

1. (20pts.) The electric potential of a spherically symmetric charge distribution is given by

$$V(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}$$

where ε_0 is the dielectric constant of the vacuum, $r = |\boldsymbol{r}|$ and $\boldsymbol{r}$ is the position vector, and q and α are constants.

(a) Find the corresponding electric field $\boldsymbol{E}(\boldsymbol{r})$ everywhere except the point $r = 0$.

(b) What is the electric charge enclosed in a sphere of radius R centered at $r = 0$? What is the total charge of the charge distribution?

(c) Find the electric charge density $\rho(\boldsymbol{r})$ everywhere except the point $r = 0$.

(d) There is a point charge at the center of the distribution. Find the charge quantity of the point charge.

(Hint: In the spherical coordinates, $\text{grad}(\varphi) = \frac{\partial\varphi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\boldsymbol{e}_\phi$;

$\text{div}(\boldsymbol{f}) = \frac{1}{r^2}\frac{\partial\left(r^2 f_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta f_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$, where $\boldsymbol{f} = f_r\boldsymbol{e}_r + f_\theta\boldsymbol{e}_\theta + f_\phi\boldsymbol{e}_\phi$.)

Solution: (a)

$$\boldsymbol{E}(\boldsymbol{r}) = -\text{grad}V(r) = -\frac{\partial V(r)}{\partial r}\boldsymbol{e}_r = -\frac{q}{4\pi\varepsilon_0}\left(-\frac{e^{-\alpha r}}{r^2} - \frac{\alpha e^{-\alpha r}}{r}\right)\boldsymbol{e}_r = \frac{q}{4\pi\varepsilon_0 r}\left(\frac{1}{r} + \alpha\right)e^{-\alpha r}\boldsymbol{e}_r.$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla V(r) = -\frac{\partial V(r)}{\partial r}\nabla r = -\frac{q}{4\pi\varepsilon_0}\left(-\frac{e^{-\alpha r}}{r^2} - \frac{\alpha e^{-\alpha r}}{r}\right)\boldsymbol{e}_r = \frac{q}{4\pi\varepsilon_0 r}\left(\frac{1}{r} + \alpha\right)e^{-\alpha r}\boldsymbol{e}_r,$$

$$\left[\nabla r = \frac{\partial r}{\partial x}\boldsymbol{e}_x + \frac{\partial r}{\partial y}\boldsymbol{e}_y + \frac{\partial r}{\partial z}\boldsymbol{e}_z = \boldsymbol{e}_r,\quad \frac{\partial r}{\partial x} = \frac{\partial\left(\sqrt{x^2+y^2+z^2}\right)}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial\left(x^2+y^2+z^2\right)}{\partial x} = \frac{x}{r}.\right]$$

(b) Using the Gauss's law, the charge enclosed by the sphere is

$$Q_R = \varepsilon_0\oint \boldsymbol{E}\cdot d\boldsymbol{S} = E\varepsilon_0 E(R)4\pi R^2 = qR\left(\frac{1}{R} + \alpha\right)e^{-\alpha R} = q(1+\alpha R)e^{-\alpha R}.$$

The total charge of the distribution is

$$Q_{\text{total}} = \lim_{R\to+\infty} Q_R = 0.$$

(c) Using the Gauss's law in differential form, we have

$$\rho(\boldsymbol{r}) = \varepsilon_0\nabla\cdot\boldsymbol{E} = \varepsilon_0\text{div}\boldsymbol{E} = \varepsilon_0\frac{1}{r^2}\frac{\partial\left[r^2 E(r)\right]}{\partial r} = \frac{q}{4\pi r^2}\frac{\partial}{\partial r}\left[(1+\alpha r)e^{-\alpha r}\right]$$

$$= \frac{q}{4\pi r^2}\left[\alpha e^{-\alpha r} - \alpha(1+\alpha r)e^{-\alpha r}\right] = -\frac{q\alpha^2}{4\pi r}e^{-\alpha r}.$$

(d) Using the result of (c), we obtain

$$Q_{r\neq 0} = \int_0^{+\infty}\rho(r)4\pi r^2 dr = -q\alpha^2\int_0^{+\infty} r e^{-\alpha r}dr = -q\int_0^{+\infty} t e^{-t}dt = q\int_0^{+\infty} t\,d\left(e^{-t}\right) = qte^{-t}\Big|_0^{+\infty} - q\int_0^{+\infty} e^{-t}dt = 0 - q = -q.$$

Thus

$$Q_{\text{point}} = Q_{\text{total}} - Q_{r\neq 0} = q.$$

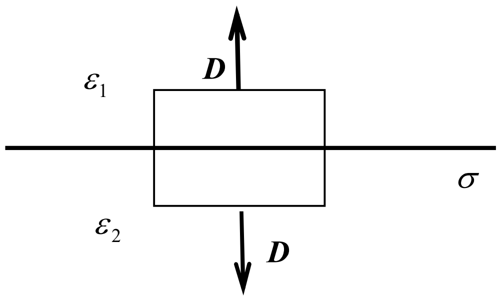
2.(20pts.) There is an infinite sheet of uniform areal charge density σ located at the plane $z=0$ in the space. The upper half space $z>0$ is filled with dielectrics of permittivity (or dielectric constant) ε_1 and the lower half space $z<0$ is filled with dielectrics of permittivity ε_2.

(a) Find the electric displacement $\boldsymbol{D}$ in both upper half ($z>0$) and lower half ($z<0$) spaces.

(b) What is the electric field $\boldsymbol{E}$ in both upper half ($z>0$) and lower half ($z<0$) spaces?

Solution: (a) Making a cylindrical Gaussian surface of base area A and placing it so that the charged sheet is centered inside the cylinder. By using the Gauss's law for electric displacement, we have

$$2DA=\sigma A \quad \text{or} \quad D=\frac{\sigma}{2}.$$



Thus

$$\boldsymbol{D}=\begin{cases}\frac{\sigma}{2}\boldsymbol{e}_z, & z>0,\\ -\frac{\sigma}{2}\boldsymbol{e}_z & z<0.\end{cases}$$

(b) The electric field is

$$\boldsymbol{E}=\begin{cases}\frac{\sigma}{2\varepsilon_1}\boldsymbol{e}_z, & z>0,\\ -\frac{\sigma}{2\varepsilon_2}\boldsymbol{e}_z & z<0.\end{cases}$$

3. (20pts.) There are two concentric conducting spherical shells of radii r_1 and r_2 ($r_1<r_2$) respectively. The inner shell of radius r_1 carries an electric charge q and the outer shell of radius r_2 carries an electric charge $-q$.

(a) What is the self-energy U_1 of the electric charge distributed on the inner shell only?

(b) What is the self-energy U_2 of the electric charge distributed on the outer shell only?

(c) What is the interaction energy between the charges on the two shells? What is the total electric potential energy of the system?

(d) Find the electric capacitance of the configuration of two conducting shells through the expression of the total electric potential energy obtained in (c).

(Hint: Here the self-energy is the potential energy of a charge distribution in the electric potential produced by the charge distribution itself. The energy is actually located in the electric field produced by the electric charge distribution and can be calculated by integral of the energy density of the electric field.)

Solution: (a) With the electric charge distributed on the inner shell only, the electric field is

$$\boldsymbol{E}_1=\begin{cases}0, & r<r_1,\\ \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{e}_r}{r^2} & r<r_1.\end{cases}$$

Thus, the energy density of the electric field in the space is

$$u_1=\frac{\varepsilon_0}{2}\boldsymbol{E}_1\cdot\boldsymbol{E}_1=\begin{cases}0, & r<r_1,\\ \frac{q^2}{32\pi^2\varepsilon_0}\frac{1}{r^4}, & r<r_1.\end{cases}$$

The self-energy is then

$$U_1=\int u_1 d\tau=\int_{r_1}^{+\infty}\frac{q^2}{32\pi^2\varepsilon_0}\frac{1}{r^4}4\pi r^2dr=\frac{q^2}{8\pi\varepsilon_0}\int_{r_1}^{+\infty}\frac{1}{r^2}dr=\frac{q^2}{8\pi\varepsilon_0}\frac{1}{r_1}.$$

(b) With the electric charge distributed on the outer shell only, the electric field is

$$\boldsymbol{E}_2=\begin{cases}0, & r<r_2,\\ -\frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{e}_r}{r^2} & r<r_2.\end{cases}$$

Thus, the energy density of the electric field in the space is

$$u_2=\frac{\varepsilon_0}{2}\boldsymbol{E}_2\cdot\boldsymbol{E}_2=\begin{cases}0, & r<r_2,\\ \frac{q^2}{32\pi^2\varepsilon_0}\frac{1}{r^4}, & r>r_2.\end{cases}$$

The self-energy is then

$$U_2 = \int u_2 d\tau = \int_{r_2}^{+\infty} \frac{q^2}{32\pi^2\varepsilon_0}\frac{1}{r^4}4\pi r^2 dr = \frac{q^2}{8\pi\varepsilon_0}\int_{r_2}^{+\infty}\frac{1}{r^2}dr = \frac{q^2}{8\pi\varepsilon_0}\frac{1}{r_2}.$$

(c) The total electric field is $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ and the total energy density is

$$u = \frac{\varepsilon_0}{2}\boldsymbol{E}\cdot\boldsymbol{E} = \frac{\varepsilon_0}{2}(\boldsymbol{E}_1+\boldsymbol{E}_2)\cdot(\boldsymbol{E}_1+\boldsymbol{E}_2) = \frac{\varepsilon_0}{2}\boldsymbol{E}_1\cdot\boldsymbol{E}_1 + \frac{\varepsilon_0}{2}\boldsymbol{E}_2\cdot\boldsymbol{E}_2 + \varepsilon_0\boldsymbol{E}_1\cdot\boldsymbol{E}_2$$
$$= u_1 + u_2 + \varepsilon_0\boldsymbol{E}_1\cdot\boldsymbol{E}_2.$$

The last term in the about equation is due to the interaction of two charge. Thus

$$u_{\text{int}} = \varepsilon_0\boldsymbol{E}_1\cdot\boldsymbol{E}_2.$$

And

$$U_{\text{int}} = \int u_{\text{int}}d\tau = \int \varepsilon_0\boldsymbol{E}_1\cdot\boldsymbol{E}_2 d\tau = \int_{r_2}^{+\infty} -\frac{q^2}{16\pi^2\varepsilon_0}\frac{1}{r^4}4\pi r^2 dr = -\frac{q^2}{4\pi\varepsilon_0}\int_{r_2}^{+\infty}\frac{1}{r^2}dr = -\frac{q^2}{4\pi\varepsilon_0}\frac{1}{r_2}.$$

The total electric potential energy is

$$U = \int u d\tau = U_1 + U_2 + U_{\text{int}} = \frac{q^2}{8\pi\varepsilon_0}\frac{1}{r_1} + \frac{q^2}{8\pi\varepsilon_0}\frac{1}{r_2} - \frac{q^2}{4\pi\varepsilon_0}\frac{1}{r_2} = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

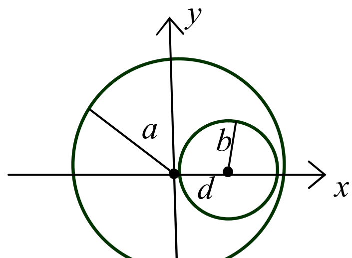
(d) Using $U = \dfrac{q^2}{2C}$, we have

$$\frac{1}{C} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right), \quad \text{and} \quad C = 4\pi\varepsilon_0\frac{r_1r_2}{r_2-r_1}.$$

4. (20pts.) An infinite long cylindrical conductor of radius a whose axis is coincident with the z-axis carries an electric current I that flows in the positive z-direction and its current density is uniform across the cross-section.

(a) Show that the magnetic induction $\boldsymbol{B}$ at cylindrical polar coordinates (ρ,ϕ,z) within the conductor $(\rho < a)$ can be written as

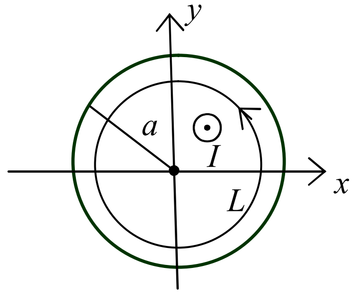
$$\boldsymbol{B} = \frac{\mu_0}{2}\frac{I\rho}{\pi a^2}\boldsymbol{e}_\phi = \frac{\mu_0}{2}\frac{I}{\pi a^2}\boldsymbol{e}_z\times\boldsymbol{\rho}$$



where $\boldsymbol{\rho} = \rho\boldsymbol{e}_\rho$, and $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_z$ are 3 orthogonal unit vectors (bases) of cylindrical coordinates.

(b)There is an infinite long cylindrical cavity of radius b $(b < a)$ in the cylindrical conductor and the axis of the cavity that is parallel to the z-axis is located at $\boldsymbol{\rho} = d\boldsymbol{e}_x$. Given that the electric current density across the cross-section with a hole is just the same as the electric current density across the hole-less cross-section in (a), show that the magnetic induction within the cavity is

$$\boldsymbol{B} = \frac{\mu_0}{2}\frac{Id}{\pi a^2}\boldsymbol{e}_y.$$



Solution: (a) Making a loop L of radius $\rho < a$ that is concentric with the cylindrical conductor. By using Ampere's circuital law, we have

$$\oint_L \boldsymbol{B}\cdot d\boldsymbol{l} = 2\pi\rho B_\phi = \mu_0\frac{I}{\pi a^2}\pi\rho^2.$$

Thus

$$B_\phi = \frac{\mu_0}{2}\frac{I\rho}{\pi a^2} \quad \text{or} \quad \boldsymbol{B} = \frac{\mu_0}{2}\frac{I\rho}{\pi a^2}\boldsymbol{e}_\phi.$$

In cylindrical coordinators, we have

$$\boldsymbol{e}_\rho\times\boldsymbol{e}_\phi = \boldsymbol{e}_z \qquad (\rho,\phi \text{ and } z \text{ form a right-handed coordinate system }).$$

A cyclic permutation gives

$$\boldsymbol{e}_\phi = \boldsymbol{e}_z\times\boldsymbol{e}_\rho.$$

Thus

$$\boldsymbol{B} = \frac{\mu_0}{2}\frac{I\rho}{\pi a^2}\boldsymbol{e}_z\times\boldsymbol{e}_\rho = \frac{\mu_0}{2}\frac{I}{\pi a^2}\boldsymbol{e}_z\times\boldsymbol{\rho}.$$

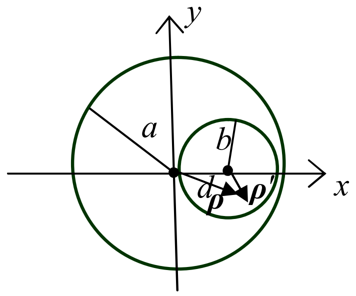
(b) The existence of the cavity in the cylindrical conductor is equivalent to imposing an electric current with equal current density but opposite in direction in the cavity region. The magnetic field produced by the opposite electric current in the cavity region can be written as

$$\boldsymbol{B}' = -\frac{\mu_0}{2}\frac{I}{\pi a^2}\boldsymbol{e}_z \times \boldsymbol{\rho}'$$

where $\boldsymbol{\rho}' = \rho' \boldsymbol{e}_{\rho'}$ and ρ' is the axial distance from the symmetric axis of the cavity. Thus

$$\boldsymbol{B}_{\text{cavity}} = \boldsymbol{B} + \boldsymbol{B}' = \frac{\mu_0}{2}\frac{I}{\pi a^2}\boldsymbol{e}_z \times (\boldsymbol{\rho} - \boldsymbol{\rho}') = \frac{\mu_0}{2}\frac{I}{\pi a^2}\boldsymbol{e}_z \times d\boldsymbol{e}_x = \frac{\mu_0}{2}\frac{Id}{\pi a^2}\boldsymbol{e}_y .$$

<u>Spherical (Polar) Coodinates</u> (r,θ,ϕ)
r : radial distance or radial coordinate
θ : polar angle or zenith
ϕ : azimuth or azimuthal angle
<u>Cylindrical (Polar) Coordinates</u> (ρ,ϕ,z)
ρ : axial distance or radial distance
ϕ : azimuth or azimuthal angle
z : axial coordinate or height



5. (20pts.) Suppose, according to Dirac's hypothesis, there are two magnetic monopoles with magnetic charges g and $2g$ ($g > 0$) that are distributed at $x = -a$ and $x = a$ on the x-axis, respectively. A magnetic dipole of moment $\boldsymbol{m} = m\boldsymbol{e}_x$ is placed at the origin $\boldsymbol{r} = 0$.
(a) What is the magnetic field $\boldsymbol{B}(x, y, z)$ produced by the two magnetic monopoles?
(b) What is potential energy of the dipole in the field of the two monopoles?
(c) What is the force $\boldsymbol{F}$ acted on the dipole?
Solution: (a) The magnetic field is

$$\boldsymbol{B} = \frac{\mu_0 g}{4\pi}\frac{\boldsymbol{r} + a\boldsymbol{e}_x}{\left|\boldsymbol{r} + a\boldsymbol{e}_x\right|^3} + \frac{2\mu_0 g}{4\pi}\frac{\boldsymbol{r} - a\boldsymbol{e}_x}{\left|\boldsymbol{r} - a\boldsymbol{e}_x\right|^3}$$

$$= \frac{\mu_0 g}{4\pi}\frac{(x+a)\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z}{\left[(x+a)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + \frac{2\mu_0 g}{4\pi}\frac{(x-a)\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z}{\left[(x-a)^2 + y^2 + z^2\right]^{\frac{3}{2}}}.$$

(b) The magnetic field at the origin is

$$\boldsymbol{B}(0,0,0) = \frac{\mu_0 g}{4\pi}\frac{a\boldsymbol{e}_x}{a^3} + \frac{2\mu_0 g}{4\pi}\frac{(-a\boldsymbol{e}_x)}{a^3} = -\frac{\mu_0 g}{4\pi a^2}\boldsymbol{e}_x .$$

The potential energy of the dipole is

$$U = -\boldsymbol{m}\cdot\boldsymbol{B}(0,0,0) = \frac{\mu_0 g m}{4\pi a^2}.$$

(c) If the dipole is placed at the coordinates (x, y, z), the potential energy is

$$U = -\boldsymbol{m}\cdot\boldsymbol{B}(x,y,z)$$

$$= -\frac{\mu_0 g m}{4\pi}\frac{x+a}{\left[(x+a)^2 + y^2 + z^2\right]^{\frac{3}{2}}} - \frac{2\mu_0 g m}{4\pi}\frac{x-a}{\left[(x-a)^2 + y^2 + z^2\right]^{\frac{3}{2}}}.$$

Thus, the force acted on the dipole

$$\boldsymbol{F} = -\nabla U\big|_{\boldsymbol{r}=0} = -\left.\frac{\partial U}{\partial x}\right|_{x=y=z=0}\boldsymbol{e}_x$$

$$= \frac{\mu_0 g m}{4\pi}\left[\frac{1}{\left[(x+a)^2 + y^2 + z^2\right]^{\frac{3}{2}}} - \frac{3(x+a)^2}{\left[(x+a)^2 + y^2 + z^2\right]^{\frac{5}{2}}} + \frac{2}{\left[(x+a)^2 + y^2 + z^2\right]^{\frac{3}{2}}} - \frac{6(x-a)^2}{\left[(x-a)^2 + y^2 + z^2\right]^{\frac{5}{2}}}\right]_{x=y=z==0}\boldsymbol{e}_x$$

$$= -\frac{6\mu_0 g m}{4\pi a^3}\boldsymbol{e}_x .$$

6. (20pts.) A monochrome plane electromagnetic wave in free space (i.e., in the vacuum) can be expressed as

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad \boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

where $\omega = c|\boldsymbol{k}| = ck$ and c is the speed of light in the vacuum.

(a) Show, by applying Maxwell equations, that

$$\boldsymbol{E} = -\frac{c^2}{\omega}\boldsymbol{k}\times\boldsymbol{B}.$$

(b) In free space, there are two travelling electromagnetic waves whose magnetic fields are

$$\boldsymbol{B}_1 = A\boldsymbol{e}_y e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)}, \quad \boldsymbol{B}_2 = A\boldsymbol{e}_y e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)}$$

where A is a constant, $\boldsymbol{k}_1 = k(\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_z)$ and $\boldsymbol{k}_2 = k(-\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_z)$. What are the corresponding electric fields? What is the Poynting vector (energy current density) of the resultant wave?

(c) What is the energy passing through the plane $z = 0$ per unit area per unit time?

Solution: Maxwell equations in free space ($\rho = 0, \boldsymbol{j} = 0$) are

$$\nabla\cdot\boldsymbol{E} = 0; \quad \nabla\cdot\boldsymbol{B} = 0; \quad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}; \quad \nabla\times\boldsymbol{B} = \varepsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t}.$$

For the monochrome plane electromagnetic wave, we have

$$\frac{\partial\boldsymbol{E}}{\partial t} = \frac{\partial}{\partial t}\left[\boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right] = \boldsymbol{E}_0\frac{\partial\left[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]}{\partial t} = -i\omega\left[\boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right] = -i\omega\boldsymbol{E},$$

and

$$\nabla\times\boldsymbol{B} = \nabla\times\left[\boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right] = \nabla\left[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]\times\boldsymbol{B}_0 = i\boldsymbol{k}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\times\boldsymbol{B}_0 = i\boldsymbol{k}\times\boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = i\boldsymbol{k}\times\boldsymbol{B}.$$

($$\nabla\times(\boldsymbol{f}\varphi) = (\nabla\times\boldsymbol{f})\varphi + (\nabla\varphi)\times\boldsymbol{f},$$

$$\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \frac{\partial\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right)}{\partial(i\boldsymbol{k}\cdot\boldsymbol{r})}\nabla(i\boldsymbol{k}\cdot\boldsymbol{r}) = ie^{i\boldsymbol{k}\cdot\boldsymbol{r}}\nabla(\boldsymbol{k}\cdot\boldsymbol{r}) = i\boldsymbol{k}e^{i\boldsymbol{k}\cdot\boldsymbol{r}},$$

$$\nabla(\boldsymbol{k}\cdot\boldsymbol{r}) = \left(\boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial r}{\partial z}\right)(k_x x + k_y y + k_z z) = k_x\boldsymbol{e}_x + k_y\boldsymbol{e}_y + k_z\boldsymbol{e}_z = \boldsymbol{k}.$$)

Substituting above two results into the fourth equation, we have

$$i\boldsymbol{k}\times\boldsymbol{B} = \varepsilon_0\mu(-i\omega_0\boldsymbol{E}) = -i\frac{\omega}{c^2}\boldsymbol{E} \qquad \text{or} \qquad \boldsymbol{E} = -\frac{c^2}{\omega}\boldsymbol{k}\times\boldsymbol{B}.$$

(c) Using the result of (a), the corresponding electric fields are

$$\boldsymbol{E}_1 = -\frac{c^2}{\omega}\boldsymbol{k}_1\times\boldsymbol{B}_1 = -A\frac{c^2}{\omega}(\boldsymbol{k}_1\times\boldsymbol{e}_y)e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)} = -A\frac{c^2k}{\omega}(\sin\theta\boldsymbol{e}_z - \cos\theta\boldsymbol{e}_x)e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)}$$
$$= -cA(\sin\theta\boldsymbol{e}_z - \cos\theta\boldsymbol{e}_x)e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)};$$

$$\boldsymbol{E}_2 = -\frac{c^2}{\omega}\boldsymbol{k}_2\times\boldsymbol{B}_2 = -A\frac{c^2}{\omega}(\boldsymbol{k}_2\times\boldsymbol{e}_y)e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)} = -A\frac{c^2k}{\omega}(-\sin\theta\boldsymbol{e}_z - \cos\theta\boldsymbol{e}_x)e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)}$$
$$= cA(\sin\theta\boldsymbol{e}_z + \cos\theta\boldsymbol{e}_x)e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)}.$$

The actual resultant electric and magnetic fields are

$$\mathrm{Re}(\boldsymbol{E}_1 + \boldsymbol{E}_2) = cA(\cos\theta\boldsymbol{e}_x - \sin\theta\boldsymbol{e}_z)\cos(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t) + cA(\cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_z)\cos(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t),$$
$$\mathrm{Re}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = A\boldsymbol{e}_y\cos(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t) + A\boldsymbol{e}_y\cos(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t).$$

The Poynting vector is

$$\boldsymbol{S} = \frac{1}{\mu_0}\mathrm{Re}(\boldsymbol{E}_1 + \boldsymbol{E}_2)\times\mathrm{Re}(\boldsymbol{B}_1 + \boldsymbol{B}_2)$$
$$= \frac{cA^2}{\mu_0}(\cos\theta\boldsymbol{e}_z - \sin\theta\boldsymbol{e}_x)\cos^2(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t) + \frac{cA^2}{\mu_0}(\cos\theta\boldsymbol{e}_z + \sin\theta\boldsymbol{e}_x)\cos^2(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t)$$
$$+ \frac{2cA^2}{\mu_0}\cos\theta\boldsymbol{e}_z\cos(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t)\cos(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t).$$

(d) The energy passing through the plane $z = 0$ per unit area per unit time is given by

$$
\begin{aligned}
I &= \boldsymbol{S} \cdot \boldsymbol{e}_z \big|_{z=0} \\
&= \frac{cA^2 \cos\theta}{\mu_0} \left[\cos^2(\boldsymbol{k}_1 \cdot \boldsymbol{r} - \omega t) + \cos^2(\boldsymbol{k}_2 \cdot \boldsymbol{r} - \omega t) + 2\cos\theta \boldsymbol{e}_z \cos(\boldsymbol{k}_1 \cdot \boldsymbol{r} - \omega t)\cos(\boldsymbol{k}_2 \cdot \boldsymbol{r} - \omega t)\right]_{z=0} \\
&= \frac{cA^2 \cos\theta}{\mu_0} \left[\cos^2(kx\sin\theta - \omega t) + \cos^2(kx\sin\theta + \omega t) + 2\cos(kx\sin\theta - \omega t)\cos(kx\sin\theta + \omega t)\right]
\end{aligned}
$$