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Name: ______________________________ November 19, 2017 Sunday

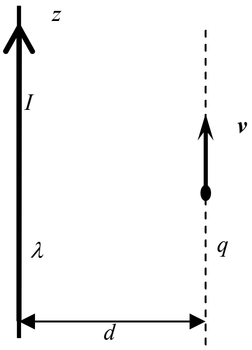
Department/School:____________________ SID:__________________________________

University Physics II

Midterm Examination

Kuang Yaming Honors School, Nanjing University

<u>Select five out of following six problems.</u>



1. (20pts.) As shown in the accompanying figure, an infinitely long straight wire with a uniform linear charge density λ lies along the z-axis. The wire also carries an electric current I that flows in the positive direction of z-axis.

(a) What is the electric field $\boldsymbol{E}$ around the wire?

(b) What is the magnetic field $\boldsymbol{B}$ around the wire?

(c) A particle of charge q is a distance d away from the wire and is travelling in a straight line parallel to the wire. What is the velocity of the particle?

Solution: (a) Making a cylindrical Gaussian surface of radius ρ and length l that is concentric with the wire, the Gauss' law gives

$$\varepsilon_0 2\pi\rho l E_\rho = \lambda\rho .$$

Thus $$E_\rho = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{\rho}, \qquad \text{or} \qquad \boldsymbol{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{\rho}\boldsymbol{e}_\rho .$$

(b) Making a loop of radius ρ that is concentric with the wire, by using the Ampere's circuital law, we have

$$2\pi\rho B_\phi = \mu_0 I .$$

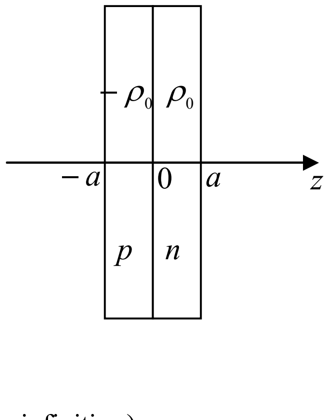
Thus $$B_\phi = \frac{\mu_0 I}{2\pi\rho} \quad \text{or} \quad \boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho}\boldsymbol{e}_\phi .$$

(c) Denoting the velocity of the particle as $\boldsymbol{v} = v\boldsymbol{e}_z$ the force acted on the charged particle is

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B} = \frac{q\lambda}{2\pi\varepsilon_0\rho}\boldsymbol{e}_\rho + \frac{\mu_0 qvI}{2\pi\rho}\boldsymbol{e}_z\times\boldsymbol{e}_\phi = \frac{q}{2\pi\rho}\left(\frac{\lambda}{\varepsilon_0} - \mu_0 vI\right)\boldsymbol{e}_\rho ..$$

The fact that particle is moving in a straight line parallel to wire implies that $\boldsymbol{F} = 0$. Thus

$$\frac{\lambda}{\varepsilon_0} - \mu_0 vI = 0 \quad \text{or} \quad v = \frac{\lambda}{\varepsilon_0\mu_0 I} = \frac{\lambda c^2}{I}.$$



2.(20pts.) When two slabs of p-type and n-type semiconductors are put in contact, the charge carriers in the p-region (holes with positive charge) tend to diffuse to n-region and the charge carriers in the n-region(electron with negative charge) tend to diffuse to the p-region. Therefore, at the interface of two types of semiconductors, an inversion layer is formed in which a volume in the n-type material is positively charged while a volume in the p-type material is negatively charged. Assume that electric charge density in the inversion layer is

$$\rho(x,y,z) = \begin{cases} \rho_0 & 0 < z < a \\ -\rho_0 & -a < z \le 0 \\ 0 & |z| \ge a \end{cases} .$$

(a) Find the electric field everywhere.

(b) By setting the zero potential point at $z = 0$, find the electric potential everywhere.

(Hint: The inversion layer is thin enough so that the areas of slabs can be approximately taken as infinities.)

Solution: (a) Obviously, $E = 0$ for $|z| > a$ .Make a cylindrical Gaussian surface with two bases parallel to $z = 0$. For $0 < z < a$, let one base be at $0 < z < a$ and the other base at $z > a$. Using the Gauss' law, we have

$$-\varepsilon_0 ES = \rho_0(a - z)S$$

Thus $E = \frac{\rho_0}{\varepsilon_0}(z-a)$.

For $-a < z \le 0$, let one base be at $-a < z \le 0$ and the other base at $z < -a$. Using the Gauss' law, we have

$$\varepsilon_0 ES = -\rho_0 (z+a)S .$$

Thus $E = -\frac{\rho_0}{\varepsilon_0}(z+a)$.

Therefore
$$\boldsymbol{E}(x,y,z) = \begin{cases} \frac{\rho_0}{\varepsilon_0}(z-a)\boldsymbol{e}_z & 0 < z < a \\ -\frac{\rho_0}{\varepsilon_0}(z+a)\boldsymbol{e}_z & -a < z \le 0 \\ 0 & |z| \ge a \end{cases} .$$

(b) $$V = -\int_{z=0}^{z} \boldsymbol{E}\cdot d\boldsymbol{l} .$$

For $0 < z < a$, $V(z) = -\int_0^z \frac{\rho_0}{\varepsilon_0}(z-a)dz = -\frac{\rho_0}{2\varepsilon_0}\left(z^2 - 2az\right)$

For $-a < z \le 0$ $V(z) = -\int_0^z -\frac{\rho_0}{\varepsilon_0}(z+a)dz = \frac{\rho_0}{2\varepsilon_0}\left(z^2 + 2az\right)$

For $z \ge a$, $V(z) = -\int_0^a \frac{\rho_0}{\varepsilon_0}(z-a)dz - \int_a^z 0dz = \frac{\rho_0}{2\varepsilon_0}a^2$.

For $z \le -a$ $V(z) = -\int_0^{-a} -\frac{\rho_0}{\varepsilon_0}(z+a)dz - \int_{-a}^{0} 0dz = -\frac{\rho_0}{2\varepsilon_0}a^2$.

Therefore

$$V(z) = \begin{cases} -\frac{\rho_0}{2\varepsilon_0}a^2 & z \le -a \\ \frac{\rho_0}{2\varepsilon_0}\left(z^2 + 2az\right) & -a < z \le 0 \\ -\frac{\rho_0}{2\varepsilon_0}\left(z^2 - 2az\right) & 0 < z < a \\ \frac{\rho_0}{2\varepsilon_0}a^2 & z \ge a \end{cases} .$$

3. (20pts.) Two square parallel-plate electrodes of sides l are a distance d apart. A dielectric slab of permittivity ε, length l, width b $(b < l)$ and thickness d is inserted between the plates, with three of four side surfaces being exactly coincident with the three of four edges of the electrodes.

(a) Find the capacitance C of the system.

(b) Find the electrostatic energy U stored in the system after the electrodes are connected to a battery of voltage V for a long enough time.

(c) The capacitor is charged as in (b) and then the battery is disconnected. What is the force acted on one of the plate now? Is the force attractive or repulsive one?

Solution: (a) The system can be treated as two capacitor connected in parallel. Therefore, the capacitance is

$$C = C_1 + C_2 \qquad C_1 = \frac{\varepsilon_0 l(l-b)}{d}, \quad C_2 = \frac{\varepsilon lb}{d} .$$

(b) The energy stored is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_0 l(l-b) + \varepsilon lb}{d}V^2 .$$

(c) After disconnected, The charge on the plate is $Q = CV = \dfrac{\varepsilon_0 l(l-b)+\varepsilon lb}{d}V$.

The energy stored can be also written as

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{d}{\varepsilon_0 l(l-b)+\varepsilon lb}Q^2 .$$

Therefore, the force acted on one of the plate is

$$F = -\left(\frac{\partial U}{\partial d}\right)_Q = -\frac{1}{2}\frac{Q^2}{\varepsilon_0 l(l-b)+\varepsilon lb} = -\frac{1}{2}\frac{\varepsilon_0 l(l-b)+\varepsilon lb}{d^2}V^2$$

It is attractive.

4. (20pts.) Suppose, according to Dirac's hypothesis, there exists a sphere of radius R that is fulfilled uniformly with amount of magnetic charge (magnetic monopole) G.

(a) Find the magnetic induction $\boldsymbol{B}$, both outside and inside the sphere.

(b) Find the energy of the magnetic field.

Solution: (a) For $r > R$, making a spherical Gaussian surface, the Gauss' law can be written as

$$\oint \boldsymbol{B}\cdot d\boldsymbol{S} = 4\pi r^2 B_r = \mu_0 G .$$

Thus $B_r = \dfrac{\mu_0 G}{4\pi r^2}$ or $\boldsymbol{B} = \dfrac{\mu_0 G}{4\pi r^2}\boldsymbol{e}_r$ $(r > R)$.

For $r < R$, making a spherical Gaussian surface, the Gauss' law can be written as

$$\oint \boldsymbol{B}\cdot d\boldsymbol{S} = 4\pi r^2 B_r = \mu_0 \frac{G}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{\mu_0 G r^3}{R^3}.$$

Thus $B_r = \dfrac{\mu_0 G r}{4\pi R^3}$ or $\boldsymbol{B} = \dfrac{\mu_0 G}{4\pi R^3}\boldsymbol{r}$ $(r > R)$.

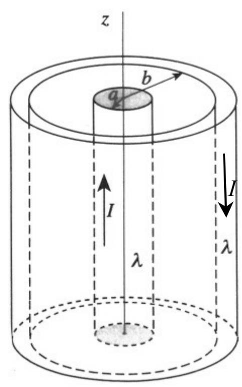
(b) The energy density of the magnetic field is

$$u = \frac{B^2}{2\mu_0} = \begin{cases} \dfrac{\mu_0 r^2 G^2}{32\pi^2 R^6} & r < R \\ \dfrac{\mu_0 G^2}{32\pi^2 r^4} & r > R \end{cases}.$$

The energy of the field is

$$U = \int u\,d\tau = \int_0^R \frac{\mu_0 r^2 G^2}{32\pi^2 R^6}4\pi r^2 dr + \int_R^\infty \frac{\mu_0 G^2}{32\pi^2 r^4}4\pi r^2 dr = \frac{\mu_0 G^2}{8\pi R^6}\int_0^R r^4 dr + \frac{\mu_0 G^2}{8\pi}\int_R^\infty \frac{1}{r^2}dr = \frac{3\mu_0 G^2}{20\pi R}.$$

4. (20pts.) The infinitely long coaxial cable carries a steady electric current I upward in the inner conductor of radius a and a return current I downward in the out conducting shell of inner radius b, as shown in the figure. The resistances per unit length of two conductors are both λ. The space between two conductors $(a < \rho < b)$ is filled with vacuum. The electric potential in the vacuum space $(a < \rho < b)$ is calculated as



$$V(\rho, z) = -I\lambda z\frac{\ln\left(\dfrac{\rho^2}{ab}\right)}{\ln\left(\dfrac{a}{b}\right)}$$

(a) Find the electric field $\boldsymbol{E}$ in the region $a < \rho < b$.

(b) Find the magnetic field $\boldsymbol{H}$ in the region $a < \rho < b$.

(c) Find the Poynting vector $\boldsymbol{S}$ in the region $a < \rho < b$.

(d) Calculate the energies per unit time flow into per unit length of two conductors respectively. Explain your results.

(Hint: In the cylindrical coordinates $\mathrm{grad}V = \dfrac{\partial V}{\partial \rho}\boldsymbol{e}_\rho + \dfrac{1}{\rho}\dfrac{\partial V}{\partial \phi}\boldsymbol{e}_\phi + \dfrac{\partial V}{\partial z}\boldsymbol{e}_z$.)

Solution: (a) $\boldsymbol{E} = -\mathrm{grad}V = \dfrac{2I\lambda z}{\rho \ln\left(\dfrac{a}{b}\right)}\boldsymbol{e}_\rho + \dfrac{I\lambda \ln\left(\dfrac{\rho^2}{ab}\right)}{\ln\left(\dfrac{a}{b}\right)}\boldsymbol{e}_z$.

(b) Making a circular loop that is concentric with the cable for $a < \rho < b$, the Ampere's circuital law gives

$$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = 2\pi\rho H_\phi = I.$$

Thus $H_\phi = \dfrac{I}{2\pi\rho}$ or $\boldsymbol{H} = \dfrac{I}{2\pi\rho}\boldsymbol{e}_\phi$.

(c) $$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H} = \frac{I^2\lambda z}{\pi\rho^2 \ln\left(\dfrac{a}{b}\right)}\boldsymbol{e}_z - \frac{I^2\lambda \ln\left(\dfrac{\rho^2}{ab}\right)}{2\pi\rho \ln\left(\dfrac{a}{b}\right)}\boldsymbol{e}_\rho.$$

(d) $$P_{\text{inner}} = \int \boldsymbol{S}\cdot d\boldsymbol{\Sigma} = \frac{I^2\lambda \ln\left(\dfrac{a^2}{ab}\right)}{2\pi a \ln\left(\dfrac{a}{b}\right)} 2\pi a\cdot 1 = I^2\lambda.$$

$$P_{\text{outer}} = \int \boldsymbol{S}\cdot d\boldsymbol{\Sigma} = -\frac{I^2\lambda \ln\left(\dfrac{b^2}{ab}\right)}{2\pi b \ln\left(\dfrac{a}{b}\right)} 2\pi b\cdot 1 = I^2\lambda.$$

The two powers flowing into the two conductors correspond to the Joule heating in the two conductors.

6. (20pts.) Suppose that in a neutral $(\rho = 0)$ and conducting medium, the physical quantities of electromagnetic field obey the linear constitutive relations $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, $\boldsymbol{B} = \mu\boldsymbol{H}$ and $\boldsymbol{j} = \sigma\boldsymbol{E}$ (Ohm's law in differential form), where ε, μ and σ are all positive constants.

(a) Write the four Maxwell equations of $\boldsymbol{E}$ and $\boldsymbol{B}$ only in differential forms, by using the above constitutive relations.

(b) Show that the electric field $\boldsymbol{E}$ satisfies the equation $\nabla^2\boldsymbol{E} - \mu\sigma\,\partial\boldsymbol{E}/\partial t - \mu\varepsilon\,\partial^2\boldsymbol{E}/\partial t^2 = 0$.

(b) Suppose the electric field $\boldsymbol{E}$ takes the form of wave solution $\boldsymbol{E} = E_0\boldsymbol{e}_x \exp[i(kz - \omega t)]$, where ω is the real angular frequency and k is the complex wave number. Find the equation satisfied by ω and k.

(Hint: $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$)

Solution:

(a) The Maxwell equations are

$$\nabla\cdot\boldsymbol{D} = \rho,$$
$$\nabla\cdot\boldsymbol{B} = 0,$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}.$$
$$\nabla\times\boldsymbol{H} = \boldsymbol{j} + \frac{\partial\boldsymbol{D}}{\partial t}.$$

Substituting the relations $\rho = 0$, $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, $\boldsymbol{B} = \mu\boldsymbol{H}$ and $\boldsymbol{j} = \sigma\boldsymbol{E}$ into the equations, we have

$$\nabla\cdot\boldsymbol{E} = 0$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\times\boldsymbol{B} = \mu\sigma\boldsymbol{E} + \varepsilon\mu\frac{\partial\boldsymbol{E}}{\partial t}.$$

(b) Taking rotation to both side of the Faraday's law, we have

Right side: $\nabla\times(\nabla\times\boldsymbol{E}) = \nabla(\nabla\cdot\boldsymbol{E}) - (\nabla\cdot\nabla)\boldsymbol{E} = -\nabla^2\boldsymbol{E}$

Left side: $\nabla\times\left(-\dfrac{\partial\boldsymbol{B}}{\partial t}\right) = -\dfrac{\partial}{\partial t}(\nabla\times\boldsymbol{B}) = -\dfrac{\partial}{\partial t}\left(\mu\sigma\boldsymbol{E} + \varepsilon\mu\dfrac{\partial\boldsymbol{E}}{\partial t}\right) = -\mu\sigma\dfrac{\partial\boldsymbol{E}}{\partial t} - \varepsilon\mu\dfrac{\partial^2\boldsymbol{E}}{\partial t^2}$

Thus: $$\nabla^2 \boldsymbol{E} - \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 .$$

(c)Substituting the wave solution into the equation, we have

$$\nabla^2 \boldsymbol{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_0 \boldsymbol{e}_x \exp[i(kz - \omega t)] = E_0 \boldsymbol{e}_x \frac{\partial^2}{\partial z^2} \exp[i(kz - \omega t)]$$
$$= -k^2 E_0 \boldsymbol{e}_x \exp[i(kz - \omega t)] = -k^2 \boldsymbol{E}.$$

$$\frac{\partial \boldsymbol{E}}{\partial t} = -i\omega \boldsymbol{E} , \quad \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\omega^2 \boldsymbol{E} .$$

Thus

$$\left(- k^2 + i\mu\sigma\omega + \varepsilon\mu\omega^2 \right)\boldsymbol{E} = 0 \text{ or } k^2 - i\mu\sigma\omega - \varepsilon\mu\omega^2 = 0 .$$