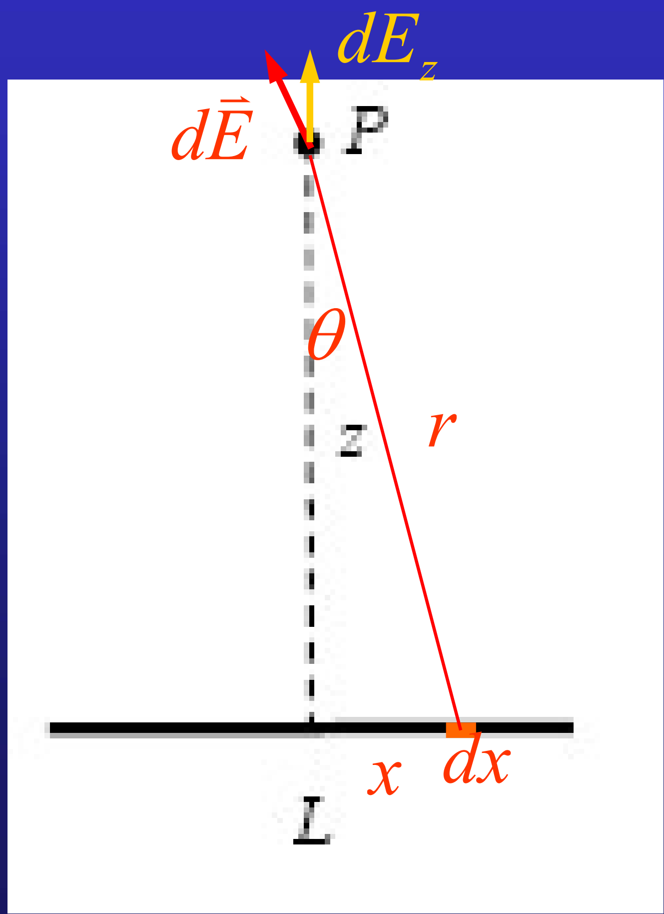
Mid-term Exam Revisited

1. (20 pts.) Find the electric field ***E*** (magnitude and direction) a distance z above the midpoint of a straight line segment of length L, which carries a uniform line charge λ. Check that your result is consistent with what you would expect when $z \gg L$.



Solution 1: (a)

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{r^2}\cos\theta$$

$$x = z\tan\theta,\ dx = z\frac{d\theta}{\cos^2\theta}$$

$$r^2 = \frac{z^2}{\cos^2\theta}$$

$$dE_z = \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{z}\cos\theta d\theta$$

$$E_z = \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{z}\int_{-\theta_0}^{\theta_0}\cos\theta d\theta = \frac{\lambda}{2\pi\varepsilon_0 z}\sin\theta_0, \qquad \tan\theta_0 = \frac{L}{2z}.$$

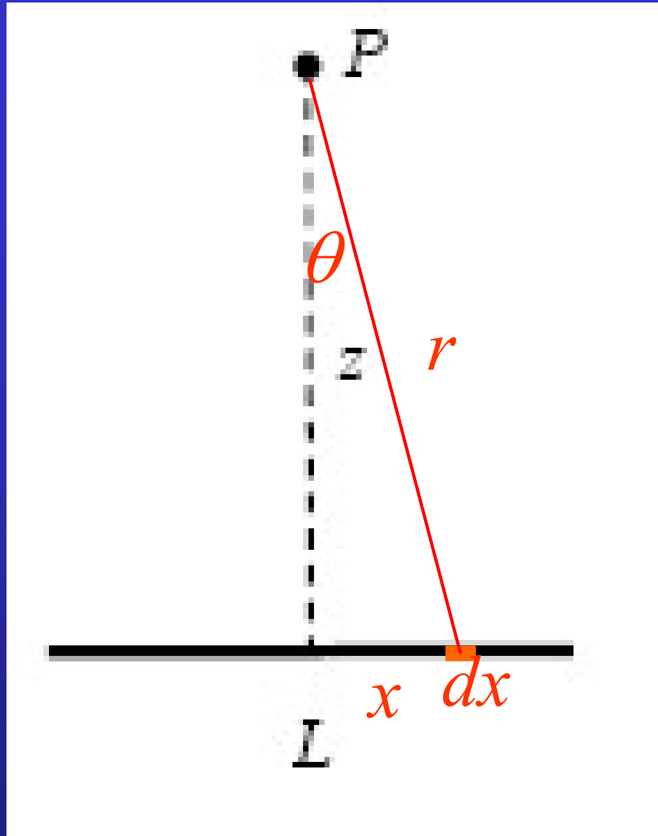
$$0 < \theta_0 \le \frac{\pi}{2}, \quad \sin\theta_0 = \frac{\tan\theta_0}{\frac{1}{\cos\theta_0}} = \frac{\tan\theta_0}{\sqrt{1+\tan^2\theta_0}} = \frac{L}{\sqrt{4z^2+L^2}}.$$

$$E_z = \frac{\lambda L}{2\pi\varepsilon_0 z\sqrt{4z^2+L^2}}, \qquad \vec{E} = \frac{\lambda L}{2\pi\varepsilon_0 z\sqrt{4z^2+L^2}}\vec{e}_z.$$

(b) When $z >> L,\quad \lambda L = q,$

$$\vec{E} = \frac{q}{2\pi\varepsilon_0 z^2}\vec{e}_z.$$

That is the electric field of a point charge.



Solution 2: (a)

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{\sqrt{x^2+z^2}}$$

Using $\displaystyle\int \frac{dx}{\sqrt{x^2+z^2}} = \ln\left|x+\sqrt{x^2+z^2}\right| + C,$

we have

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{\lambda dx}{\sqrt{x^2+z^2}} = \frac{1}{2\pi\varepsilon_0}\ln\left(\frac{L}{2z}+\sqrt{\left(\frac{L}{2z}\right)^2+1}\right).$$

Finally

$$E_z = -\frac{dV(z)}{dz} = \frac{\lambda L}{2\pi\varepsilon_0 z\sqrt{4z^2 + L^2}}.$$

$$\int \frac{dx}{\sqrt{x^2 + z^2}} = \ln\left|x + \sqrt{x^2 + z^2}\right| + C,$$

Let $\sqrt{x^2 + z^2} = t - x,$ we have $x = \frac{t^2 - z^2}{2t},$

$$\sqrt{x^2 + z^2} = t - \frac{t^2 - z^2}{2t} = \frac{t^2 + z^2}{2t}, \qquad dx = \frac{t^2 + z^2}{2t^2} dt.$$

$$\int \frac{dx}{\sqrt{x^2 + z^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|x + \sqrt{x^2 + z^2}\right| + C.$$

2. (20 pts.) Suppose that there exists a magnetic monopole g that is located at the origin and a magnetic dipole with the magnetic dipole moment of $\boldsymbol{m}$ is placed on the x-axis at $x=a$ ($a > 0$).

(a) What is the magnetic field $\boldsymbol{B}$ produced by the monopole at the position of the dipole?

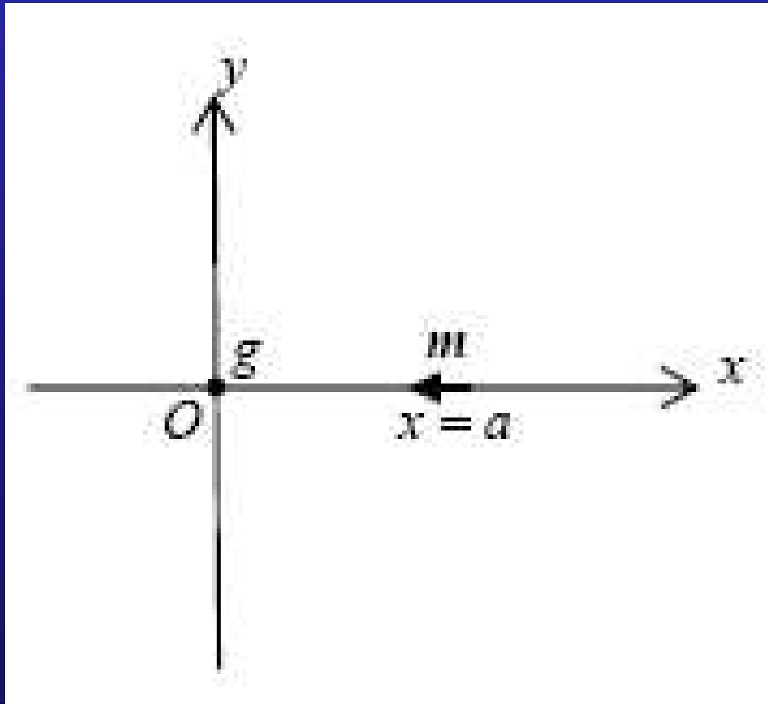
(b) What is the potential energy of the dipole in the magnetic field of the monopole?

(c) If the dipole moment $\boldsymbol{m}$ is anti-parallel to the x-axis, find the force $\boldsymbol{f}$ acted on it.

Solution 1: (a)

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{g}{r^2}\vec{e}_r.$$

$$\vec{r} = a\vec{e}_x, \qquad \vec{B} = \frac{\mu_0}{4\pi}\frac{g}{a^2}\vec{e}_x.$$



(b) $$U=-\vec{m}\cdot\vec{B}+C=-\frac{\mu_0}{4\pi}\frac{g}{a^2}\vec{m}\cdot\vec{e}_x+C=-\frac{\mu_0}{4\pi}\frac{g}{a^2}m_x+C.$$

(c) $$\vec{m}=-m\vec{e}_x \qquad U(x)=-\frac{\mu_0}{4\pi}\frac{g}{x^2}\left(-m\right)+C=\frac{\mu_0}{4\pi}\frac{gm}{x^2}+C,$$

$$f_x=-\frac{dU(x)}{dx}=\frac{\mu_0}{2\pi}\frac{gm}{x^3}, \qquad \vec{f}=\frac{\mu_0}{2\pi}\frac{gm}{x^3}\vec{e}_x.$$

3.(20 pts.) A long cylindrical shell of length L has an inner radius a and an outer radius b is made of a material of electrical conductivity σ. The inner and outer surfaces of the shell are maintained at a potential difference V. (The two cylindrical surfaces are equipotential surfaces.)

(a) What is the electric field inside the shell ($a < \rho < b$)?

(b) What is the electric current that flows from the inner surface to the outer surface?

Solution1 (b). The electrical resistance is

$$R = \int_a^b \frac{d\rho}{\sigma(2\pi\rho L)} = \frac{1}{2\pi L\sigma} \ln\frac{b}{a}.$$

The electric current is

$$I = \frac{V}{R} = \frac{2\pi L\sigma V}{\ln(b/a)}.$$

(a) The electric current density is

$$j = \frac{I}{2\pi\rho L} = \frac{\sigma V}{\rho \ln(b/a)}, \qquad \vec{j} = \frac{\sigma V}{\rho \ln(b/a)} \vec{e}_\rho .$$

By using the Ohm's law in differential form $\vec{j} = \sigma \vec{E}$, we obtain

$$\vec{E} = \frac{V}{\rho \ln(b/a)} \vec{e}_\rho .$$

Solution2: (a) Assume that the electric charge per unit length accumulated on the inner surface is λ. Making a cylindrical Gaussian surface that is concentric with the conductor shell, we have

$$\varepsilon_0 E \cdot 2\pi\rho L = \lambda L, \quad E = \frac{\lambda}{2\pi\varepsilon_0 \rho}.$$

The potential difference is then

$$V = \int_a^b E d\rho = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 \rho} = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{b}{a}.$$

Thus, we have $\lambda = \dfrac{V}{\dfrac{1}{2\pi\varepsilon_0}\ln\dfrac{b}{a}}$, and

$$E = \frac{V}{\rho\ln(b/a)}, \qquad \vec{E} = \frac{V}{\rho\ln(b/a)}\vec{e}_\rho.$$

(b) By using the Ohm's law in differential form $\vec{j} = \sigma\vec{E}$, we obtain

$$\vec{j} = \frac{\sigma V}{\rho\ln(b/a)}\vec{e}_\rho.$$

The electric current is then

$$I = jS = j2\pi\rho L = \frac{2\pi L\sigma V}{\ln(b/a)}.$$

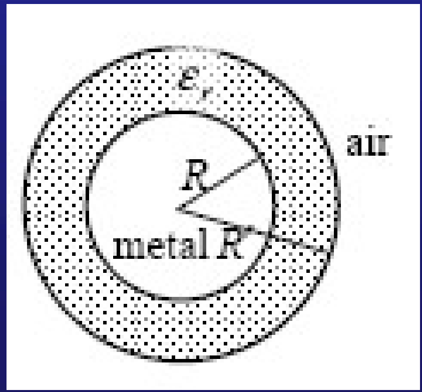
4.(20 pts.) A metal sphere of radius *R* that carries electric charges *Q* is insulated from the air with a spherical shell of medium of relative dielectric constant ε_r. The outer radius of the dielectric medium is *R*'. The relative dielectric constant of the air can be taken as 1.0 .

(a) Find the electric field ***E*** outside the metal sphere.

(b) Find the areal charge density σ on the outer surface of the dielectric shell.

(c) What is the capacitance *C* of this configuration?

Solution1 (a). Making a spherical Gaussian surface (*r*>*R*) that is concentric with the metal sphere, we have



$$4\pi r^2 D = Q, \quad D = \frac{Q}{4\pi r^2}, \qquad r > R.$$

Thus
$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_r\varepsilon_0 r^2}\vec{e}_r, & R' > r > R; \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\vec{e}_r, & r > R'. \end{cases}$$

(b) Solution1: The electric polarization of the medium is

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \frac{Q}{4\pi r^2}\left(1 - \frac{1}{\varepsilon_r}\right)\vec{e}_r.$$

Thus the areal (surface) (polarization) charge density is

$$\sigma_P = \vec{P} \cdot \vec{n} = \frac{Q}{4\pi R'^2}\left(1 - \frac{1}{\varepsilon_r}\right)\vec{e}_r \cdot \vec{e}_r = \frac{Q}{4\pi R'^2}\left(1 - \frac{1}{\varepsilon_r}\right).$$

Solution2: Making a pillbox-like Gaussian surface that is across the interface between the medium and the air, and its two bottoms are parallel to the interface. Assume the bottom area of the Gaussian surface is A . The Gauss's law can be written as

$$\varepsilon_0\left[E(R' + 0^+) - E(R' - 0^+)\right]A = \sigma_P A.$$

Thus

$$\sigma_P = \varepsilon_0\left[E(R'+0^+) - E(R'-0^+)\right]$$

$$= \varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 R'^2} - \frac{Q}{4\pi\varepsilon_r\varepsilon_0 R'^2}\right) = \frac{Q}{4\pi R'^2}\left(1 - \frac{1}{\varepsilon_r}\right).$$

(c) The electrical potential of the metal sphere is

$$V = -\int_{-\infty}^{R} E dr = -\int_{-\infty}^{R'} \frac{Q}{4\pi\varepsilon_0 r^2} dr - \int_{R'}^{R} \frac{Q}{4\pi\varepsilon_r\varepsilon_0 r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon_0 R'} + \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{1}{R} - \frac{1}{R'}\right).$$

The capacitance is

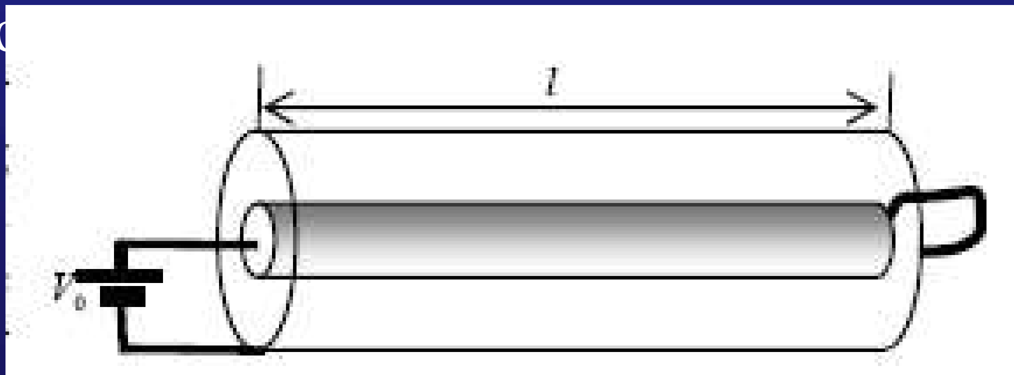
$$C = \frac{Q}{V} = 4\pi\varepsilon_r\varepsilon_0 R\left(\frac{R'}{R' + (\varepsilon_r - 1)R}\right).$$

5. (20 pts.) Consider a long coaxial cable in which the inner conductor is a cylinder of radius a made of material having electrical resistivity ρ and relative magnetic permeability μ_r and the out shield is a perfect conductor in the shape of a thin cylindrical shell of radius b. The inner conductor is connected (shorted) to the out shield at the right end and a voltage V_0 is applied at the left end of the cable (as shown in the figure) so that a steady current is established after some time. Assume that the electric current densities are uniform in the inner conductor and in the out shield of the cable and the length of the cable is large enough ($l>>b>a$) so that the edge effect can be neglected.

(a) Calculate the magnetic field everywhere within the coaxial cable.

(b) Find the energy of the magnetic field.

(c) Find the self-inductance c



Solution: (a) The resistance of the inner conductor is $R = \frac{\rho l}{\pi a^2}$ and and the electric current is

$$I = \frac{V_0}{R} = \frac{V_0 \pi a^2}{\rho l}.$$

The electric current density in the inner conductor is

$$j = \frac{I}{\pi a^2} = \frac{V_0}{\rho l}.$$

Using the Ampere's circuital law $\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I$, we obtain

$$B = \begin{cases} \dfrac{\mu_0 \mu_r V_0 r}{2\rho l}, & r < a; \\ \dfrac{\mu_0 V_0 a^2}{2\rho l r}, & a < r < b. \end{cases}$$

(b) The energy of the magnetic field is

$$U_m = \int \frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}\,d\tau = \frac{1}{2}(2\pi l)\left(\int_0^a \frac{\mu_0\mu_r V_0^2 r^2}{4\rho^2 l^2} r\,dr + \int_a^b \frac{\mu_0 V_0^2 a^4}{4\rho^2 l^2 r^2} r\,dr\right)$$

$$= \frac{\pi\mu_0\mu_r V_0^2 a^4}{16\rho^2 l} + \frac{\pi\mu_0 V_0^2 a^4}{4\rho^2 l}\ln\left(\frac{b}{a}\right)$$

(c) Rewriting the energy of the magnetic field as

$$U_m = \frac{1}{2}\left(\frac{\mu_0\mu_r l}{8\pi} + \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)\right)I^2 = \frac{1}{2}LI^2$$

we obtain that the self-inductance is

$$L = \frac{\mu_0\mu_r l}{8\pi} + \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{\mu_0 l}{2\pi}\left(\frac{\mu_r}{4} + \ln\left(\frac{b}{a}\right)\right).$$

6. (20 pts.) An electromagnetic wave travels in free space. The electric field is $\boldsymbol{E} = E_0\,[\cos(\boldsymbol{k}\bullet\boldsymbol{r} - \omega t)\,\boldsymbol{e}_x + \sin(\boldsymbol{k}\bullet\boldsymbol{r} - \omega t)\,\boldsymbol{e}_y]$, where $E_0 > 0$ is a constant, and ω and $\boldsymbol{k} = k\,\boldsymbol{e}_z$ are the frequency and wave vector ($\omega = c\,k$ where c is the speed of light), respectively. Here, x, y and z are Cartesian coordinates, $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ the corresponding unit vectors, t denotes the time, and $\boldsymbol{r} = x\,\boldsymbol{e}_x + \mathrm{y}\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z$.

(a) What is the spinning direction of $\boldsymbol{E}$ at a fixed point $\boldsymbol{r}$, clockwise or anticlockwise?

(b) Verify that the electric field $\boldsymbol{E}$ satisfies the equation

$$\nabla^2\boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} = 0.$$

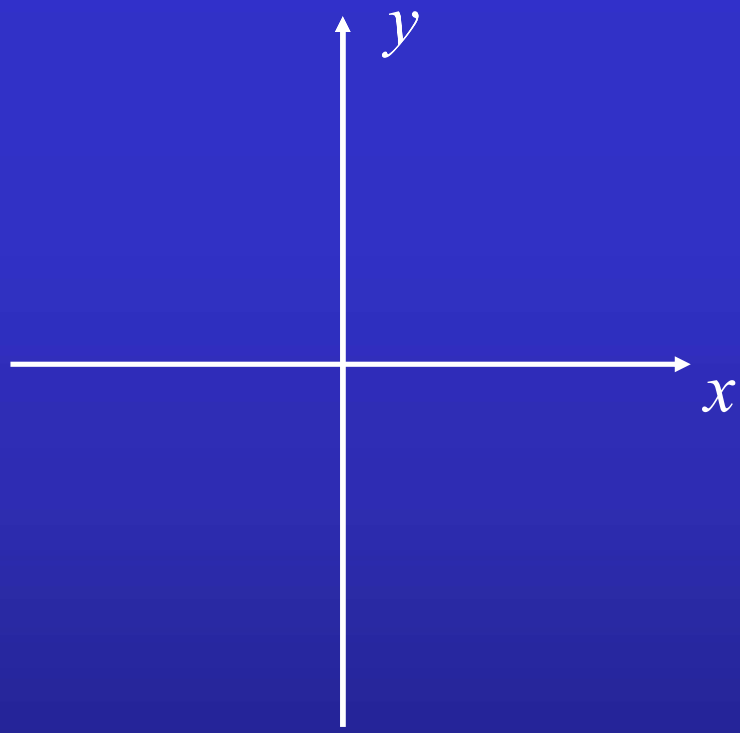
(c) Find the magnetic induction $\boldsymbol{B}$.

(d) Find the Poynting vector $\boldsymbol{S}$.

Solution: (a) It is clockwise.

$$E_x = E_0 \cos(kz - \omega t)$$

$$E_y = E_0 \sin(kz - \omega t)$$



(b) $$\nabla^2 \boldsymbol{E} = -\boldsymbol{k}^2 \boldsymbol{E},$$

$$\frac{\partial^2}{\partial^2 t} \boldsymbol{E} = -\omega^2 \boldsymbol{E},$$

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} \boldsymbol{E} = \left(-\boldsymbol{k}^2 + \frac{\omega^2}{c^2} \right) \boldsymbol{E} = 0.$$

(c)

$$\nabla\times\boldsymbol{E}=-\frac{\partial}{\partial t}\boldsymbol{B},$$

$$\boldsymbol{E}=E_0[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x+\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y],$$

$$\nabla\times\boldsymbol{E}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_0\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) & E_0\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) & 0\end{vmatrix}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_0\cos(kz-\omega t) & E_0\sin(kz-\omega t) & 0\end{vmatrix}$$

$$=-kE_0\cos(kz-\omega t)\boldsymbol{e}_x-kE_0\sin(kz-\omega t)\boldsymbol{e}_y=-kE_0[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x-\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y]$$

$$\boldsymbol{B}=B_0[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y-\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x],$$

$$\frac{\partial}{\partial t}\boldsymbol{B}=\omega B_0[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x-\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y],$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial}{\partial t}\boldsymbol{B}\quad\Rightarrow\quad B_0=\frac{1}{c}E_0.$$

$$\boldsymbol{B}=\frac{1}{c}E_0[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y-\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x].$$

(d)

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$$

$$= \frac{1}{\mu_0} E_0 \frac{1}{c} E_0 [\cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\boldsymbol{e}_x + \sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\boldsymbol{e}_y]$$

$$\times [\cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\boldsymbol{e}_y - \sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\boldsymbol{e}_x]$$

$$= \frac{E_0^2}{\mu_0 c} \boldsymbol{e}_z .$$