1	2	3	4	5	6	Total
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Name: ________________________________ November 6, 2016 Sunday

Department/School:______________________ SID:_______________________________________

University Physics II

Midterm Examination

Kuang Yaming Honors School, Nanjing University

Select five out of following six problems.

1.(20pts.) Three point charges $2q$, q and $2q$ are restricted to the closed interval $-a \le x \le a$ on the x-axis.

(a) Determine the positions where the charges should be placed so that the potential energy of the system is a minimum.

(b) Supposing that the three point charges are located at the positions as determined in (a), find the electrical potential $V(x)$ at the point x on the x-axis where $|x| >> a$, and approximate the result by expanding it up to the second order of $a/|x|$.

(c) For the same arrangement of the point charges in (b), find the approximate electrical field $\boldsymbol{E}(x)$ at the point x on the x-axis (where $|x| >> a$) up to the second order of $a/|x|$.

Solution: (a) To acquire the minimum potential energy, the two charges $2q$ should be placed as far apart as possible, i.e., one at $x = -a$ and the other at $x = a$. Then, supposing that the point charge q is placed at a point x ($-a \le x \le a$), the total potential energy is given by

$$U = \frac{1}{4\pi\varepsilon_0}\frac{(2q)^2}{2a} + \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a-x} + \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a+x} = \frac{2q^2}{4\pi\varepsilon_0}\left(\frac{1}{a} + \frac{2a}{a^2 - x^2}\right).$$

Obviously, when $x = 0$, the minimum potential energy is $U_{\min} = \dfrac{3q^2}{2\pi\varepsilon_0 a}$. Therefore, the positions of three point charges $2q$, q and $2q$ are $x = -a, 0, a$ respectively.

(b)
$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{1}{4\pi\varepsilon_0}\frac{2q}{|x-a|} + + \frac{1}{4\pi\varepsilon_0}\frac{2q}{|x+a|}.$$

$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{q}{2\pi\varepsilon_0|x|}\left(\left|\frac{1}{1-a/x}\right| + \left|\frac{1}{1+a/x}\right|\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{q}{2\pi\varepsilon_0|x|}\left(\left|1 + \frac{a}{x} + \frac{a^2}{x^2} + \cdots\right| + \left|1 - \frac{a}{x} + \frac{a^2}{x^2} + \cdots\right|\right)$$

$$\approx \frac{5q}{4\pi\varepsilon_0|x|} + \frac{qa^2}{\pi\varepsilon_0|x|^3}.$$

(c)
$$E(x) = -\frac{\partial V(x)}{\partial x} = \frac{5qx}{4\pi\varepsilon_0|x|^3} + \frac{3qa^2x}{\pi\varepsilon_0|x|^5}, \qquad \boldsymbol{E}(x) = \frac{5qx}{4\pi\varepsilon_0|x|^3}\boldsymbol{e}_x + \frac{3qa^2x}{\pi\varepsilon_0|x|^5}\boldsymbol{e}_x.$$

2.(20pts.) A pair of magnetic monopoles are placed at $\boldsymbol{r} = \boldsymbol{a}$ and $\boldsymbol{r} = -\boldsymbol{a}$, with the corresponding magnetic charges being g and $-g$, respectively.

(a) According to Dirac's hypothesis, write down the magnetic induction $\boldsymbol{B}$ at the point $\boldsymbol{r}$ ($\boldsymbol{r} \ne \pm\boldsymbol{a}$).

(b)The configuration can also be taken as a magnetic dipole when viewed from the far zone ($r >> a, a = |\boldsymbol{a}|$), and the dipole moment can be defined as $\boldsymbol{p} = 2g\boldsymbol{a}$. Find the magnetic induction $\boldsymbol{B}$ produced by this dipole by expanding the expression of (a) up to the first order of a/r.

(c) Suppose there is another magnetic dipole moment $\boldsymbol{\mu}$ that is constituted of a small electric current loop in the magnetic field $\boldsymbol{B}$ of (b). The dipole moment $\boldsymbol{\mu}$ is located at the position $\boldsymbol{r}$ within the far zone. Find the torque $\boldsymbol{M}$ acted on the magnetic dipole $\boldsymbol{\mu}$ if $\boldsymbol{r} = x\,\boldsymbol{i}$, $\boldsymbol{\mu} = \mu\,\boldsymbol{j}$ and $\boldsymbol{a} = a\,\boldsymbol{k}$ where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the three base vectors of Cartesian system.

(Hint: $f(\boldsymbol{r}+\boldsymbol{a}) = f(\boldsymbol{r}) + \boldsymbol{a}\cdot\nabla f(\boldsymbol{r}) + \cdots$, and in particular $\dfrac{1}{|\boldsymbol{r}+\boldsymbol{a}|^3} = \dfrac{1}{r^3} + \boldsymbol{a}\cdot\nabla\dfrac{1}{r^3} + \cdots$ and $\nabla\dfrac{1}{r^3} = -\dfrac{3\boldsymbol{r}}{r^5}$.)

Solution: (a) According to Dirac's hypothesis, the magnetic induction produced by two magnetic monopoles (or two point magnetic charges) is

$$\boldsymbol{B}=\frac{\mu_0}{4\pi}\frac{g}{|\boldsymbol{r}-\boldsymbol{a}|^3}(\boldsymbol{r}-\boldsymbol{a})-\frac{\mu_0}{4\pi}\frac{g}{|\boldsymbol{r}+\boldsymbol{a}|^3}(\boldsymbol{r}+\boldsymbol{a}).$$

(b)

$$\begin{aligned}\boldsymbol{B}&=\frac{\mu_0}{4\pi}\frac{g}{|\boldsymbol{r}-\boldsymbol{a}|^3}(\boldsymbol{r}-\boldsymbol{a})-\frac{\mu_0}{4\pi}\frac{g}{|\boldsymbol{r}+\boldsymbol{a}|^3}(\boldsymbol{r}+\boldsymbol{a})\\&=\frac{\mu_0}{4\pi}\left[\frac{g}{|\boldsymbol{r}-\boldsymbol{a}|^3}-\frac{g}{|\boldsymbol{r}+\boldsymbol{a}|^3}\right]\boldsymbol{r}-\frac{\mu_0}{4\pi}\left[\frac{g}{|\boldsymbol{r}+\boldsymbol{a}|^3}+\frac{g}{|\boldsymbol{r}+\boldsymbol{a}|^3}\right]\boldsymbol{a}=\frac{\mu_0}{4\pi}\left(-2g\boldsymbol{a}\cdot\nabla\frac{1}{r^3}\right)\boldsymbol{r}-\frac{\mu_0}{4\pi}\frac{2g}{r^3}\boldsymbol{a}\\&=\frac{\mu_0}{4\pi}\left(6g\boldsymbol{a}\cdot\frac{\boldsymbol{r}}{r^5}\right)\boldsymbol{r}-\frac{\mu_0}{4\pi}\frac{2g}{r^3}\boldsymbol{a}=\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\boldsymbol{p}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r-\boldsymbol{p}\right].\end{aligned}$$

(c)Substituting $\boldsymbol{r}=x\boldsymbol{i},\ \boldsymbol{p}=2ga\boldsymbol{k}$ into(b), we obtain $\boldsymbol{B}=-\frac{\mu_0}{4\pi}\frac{2ga}{x^3}\boldsymbol{k}$ .Thus

$$\boldsymbol{M}=\boldsymbol{\mu}\times\boldsymbol{B}=\mu\boldsymbol{j}\times\boldsymbol{B}=-\frac{\mu_0}{4\pi}\frac{2ga\mu}{x^3}\boldsymbol{i}.$$

3. (20pts.) Three thin cylindrical shells are coaxially fixed together; their radii are R, $2R$, and $4R$, respectively. The inner and outer shells are both metallic and are both connected to the ground; the middle one is plastic, with areal charge density σ.

(a) Find the areal charge density σ_1 of the inner shell.

(b) Find the electric field $\boldsymbol{E}_1$ between the inner and the middle shells, and $\boldsymbol{E}_2$ between the middle and the outer ones.

(a)Making a cylindrical Gaussian surface that is concentric with the cylindrical shells and its radius is $R<r<2R$, using Gauss' law, we have

$$\varepsilon_0 E\cdot 2\pi rl=\sigma_1 2\pi Rl.$$

Thus $$E=\frac{R}{r}\frac{\sigma_1}{\varepsilon_0},\quad \boldsymbol{E}=\frac{R}{r}\frac{\sigma_1}{\varepsilon_0}\boldsymbol{e}_\rho,\ (R<r<2R).$$

Making a cylindrical Gaussian surface that is concentric with the cylindrical shells and its radius is $2R<r<4R$, using Gauss' law, we have

$$\varepsilon_0 E\cdot 2\pi rl=\sigma_1 2\pi Rl+\sigma 2\pi(2R)l\ .$$

Thus $$E=\frac{R}{r}\frac{\sigma_1+2\sigma}{\varepsilon_0},\quad \boldsymbol{E}=\frac{R}{r}\frac{\sigma_1+2\sigma}{\varepsilon_0}\boldsymbol{e}_\rho,\quad (2R<r<4R).$$

The two metallic shells are equipotential, so we have

$$\begin{aligned}V(R)-V(4R)&=\int_R^{4R}Edr=\int_R^{2R}Edr+\int_{2R}^{4R}Edr=\int_R^{2R}\frac{R}{r}\frac{\sigma_1}{\varepsilon_0}dr+\int_{2R}^{4R}\frac{R}{r}\frac{\sigma_1+2\sigma}{\varepsilon_0}dr\\&=\frac{\sigma_1}{\varepsilon_0}R\ln 2+\frac{\sigma_1+2\sigma}{\varepsilon_0}R\ln 2=\frac{2(\sigma_1+\sigma)}{\varepsilon_0}R\ln 2=0.\end{aligned}$$

Therefore, we obtain $\sigma_1=-\sigma$.

(b) Finally, the electric field is

$$\boldsymbol{E}=\begin{cases}-\frac{R}{r}\frac{\sigma}{\varepsilon_0}\boldsymbol{e}_\rho, & R<r<2R,\\ \frac{R}{r}\frac{\sigma}{\varepsilon_0}\boldsymbol{e}_\rho, & 2R<r<4R.\end{cases}$$

$-\sigma$ $-\sigma$

4. (20pts.) Two parallel plates enclose a dielectric material that has a varying relative dielectric function: $\varepsilon_r = \varepsilon_r(x) = e^{\alpha x}$ where $\alpha > 0$ is a constant and x is the distance apart from the lower plate. Suppose that the areal charge densities of the upper and lower plates are σ and $-\sigma$ respectively.

(a) What is the electric displacement $\boldsymbol{D}$ within the medium?

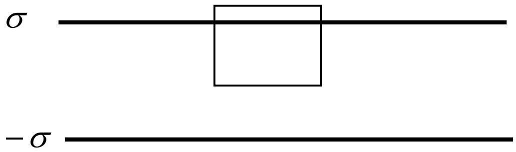
(b) What is the electric field $\boldsymbol{E}$ within the medium?

(c) If the two plates are of area S, and a distance d apart, find the energy U stored in this configuration.

(d) Find the volume charge density ρ distributing within the medium.

Solution: (a) Making a cylindrical Gaussian surface with two bases being parallel to the plates, one base outsider the upper plates and another base inside the dielectrics, using the Gauss's law, we have

$$DS = \sigma S\,,\ \ D = \sigma\,,\ \ \boldsymbol{D} = -\sigma \boldsymbol{e}_x .$$



(b) The electric field is

$$\boldsymbol{E} = \frac{1}{\varepsilon}\boldsymbol{D} = \frac{1}{\varepsilon_r\varepsilon_0}\boldsymbol{D} = -\frac{1}{\varepsilon_r\varepsilon_0}\sigma\boldsymbol{e}_x = -\frac{e^{-\alpha x}}{\varepsilon_0}\sigma\boldsymbol{e}_x .$$

(c1) The potential difference between the upper and lower plates is

$$V = -\int_0^d \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = \int_0^d \frac{e^{-\alpha x}}{\varepsilon_0}\sigma\,\mathrm{d}x = \frac{1-e^{-\alpha d}}{\varepsilon_0\alpha}\sigma .$$

Thus, the electric capacitance of the configuration is

$$C = \frac{Q}{V} = \frac{\sigma S}{\dfrac{1-e^{-\alpha d}}{\varepsilon_0\alpha}\sigma} = \varepsilon_0\alpha\frac{S}{1-e^{-\alpha d}},$$

and electric potential energy stored is

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{1-e^{-\alpha d}}{\varepsilon_0\alpha}\sigma^2 S .$$

(c2) The energy density of the electric field is

$$u = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} = \frac{1}{2\varepsilon}\boldsymbol{D}^2 = \frac{1}{2}\frac{e^{-\alpha x}}{\varepsilon_0}\sigma^2 .$$

Then, the electric potential energy is

$$U = \int_0^d u(x)S dx = \int_0^d \frac{1}{2}\frac{e^{-\alpha x}}{\varepsilon_0}\sigma^2 S dx = \frac{1}{2}\frac{1-e^{-\alpha d}}{\varepsilon_0\alpha}\sigma^2 S .$$

(d1) Since there is no free electric charge inside the medium, the electric charge inside the medium is the polarization charge(or bound charge). The electric polarization vector is

$$\boldsymbol{P} = \boldsymbol{D} - \varepsilon_0\boldsymbol{E} = -\sigma\boldsymbol{e}_x + e^{-\alpha x}\sigma\boldsymbol{e}_x = (e^{-\alpha x} - 1)\sigma\boldsymbol{e}_x .$$

The polarization charge density is

$$\rho_P = -\nabla\cdot\boldsymbol{P} = -\frac{\partial(e^{-\alpha x}-1)}{\partial x}\sigma = \alpha\sigma e^{-\alpha x} .$$

(d2) Using the Gauss's law in differential form $\nabla\cdot\boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$, we obtain

$$\rho = \varepsilon_0\nabla\cdot\boldsymbol{E} = -\sigma\nabla\cdot\left(e^{-\alpha x}\boldsymbol{e}_x\right) = -\sigma\frac{\partial}{\partial x}e^{-\alpha x} = \alpha\sigma e^{-\alpha x} .$$

Since there is no free electric charge inside the medium, the electric charge inside the medium is the polarization charge(or bound charge). The existence of the volume polarization charge density is due to the non-uniformity of the electric polarization vector.

5. (20 pts.) A long solenoid of radius a, carrying n turns per unit length, is looped perpendicularly from outside by a closed conducting wire of electrical resistance R.

(a) If the current I in the solenoid is increasing at a constant rate ($dI/dt = k$), what electrical current J flows in the loop?

(b) If the current I in the solenoid is keeping constant but the solenoid is pulled out of the wire loop, turned around, and reinserted into it perpendicularly, what total charge Q that passes through the conducting loop during the process.

Solution:

(a)The magnetic induction inside the solenoid is $B = \mu_0 nI$ and the magnetic flux passing through the conducting wire loop is

$$\Phi_m = B\pi a^2 = \mu_0 n\pi a^2 I .$$

By using Faraday's law, the electromotive force induced is

$$\varepsilon = -\frac{d\Phi_m}{dt} = \mu_0 n\pi a^2 \frac{dI}{dt} = -\mu_0 n\pi a^2 k.$$

And the electrical current flows in the loop is

$$|J| = \frac{|\varepsilon|}{R} = \frac{\mu_0 nI\pi a^2 k}{R}.$$

(b) During the process, the total change in magnetic flux is

$$\Delta\Phi_m = -\mu_0 n\pi a^2 I - \mu_0 n\pi a^2 I = -2\mu_0 n\pi a^2 I$$

Thus, total charge Q that passes through the conducting loop is

$$Q = \int_{-\infty}^{+\infty} J dt = \int_{-\infty}^{+\infty}\left(-\frac{1}{R}\frac{d\Phi_m}{dt}\right)dt$$

$$= -\frac{1}{R}\int_{-\infty}^{+\infty} d\Phi_m = -\frac{1}{R}\Delta\Phi_m = \frac{2\mu_0 nI\pi a^2}{R}.$$

6. (20pts.) In free space, there exists a traveling electromagnetic wave $\boldsymbol{B}(x, y, z) = B_0\cos(kz - \omega t)\,\boldsymbol{e}_y$ where $\boldsymbol{B}$ is the magnetic induction. Here, B_0 is a constant, x, y and z are Cartesian coordinates, and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ the corresponding unit vectors, t denotes the time, k the wave vector, and ω the frequency: $\omega = ck$ where c is the speed of light.

(a) Show that $\nabla\cdot\boldsymbol{B} = 0$.

(b) Supposing that the electric field $\boldsymbol{E}$ takes on the form: $\boldsymbol{E} = \boldsymbol{E}_0\cos(kz - \omega t)$ where $\boldsymbol{E}_0$ is a constant vector, find $\boldsymbol{E}_0$ by the Maxwell equation $\nabla\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}$.

(c) Find the Poynting vector $\boldsymbol{S}$ and its average $\langle\boldsymbol{S}\rangle$ over one period of time.

Solution: (a)

$$\nabla\cdot\boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial}{\partial y}B_0\cos(kz - \omega t) = 0.$$

(b)

$$\nabla\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & B_0\cos(kz - \omega t) & 0 \end{vmatrix} = kB_0\sin(kz - \omega t)\boldsymbol{e}_x,$$

$$\frac{\partial}{\partial t}\boldsymbol{E} = \omega\boldsymbol{E}_0\sin(kz - \omega t).$$

Substituting above expressions into the equation $\nabla\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}$, we have

$$kB_0\sin(kz - \omega t)\boldsymbol{e}_x = \frac{1}{c^2}\omega\boldsymbol{E}_0\sin(kz - \omega t).$$

Then we obtain $\boldsymbol{E}_0 = c^2\frac{k}{\omega}B_0\boldsymbol{e}_x = cB_0\boldsymbol{e}_x$. Therefore, the corresponding electric field is

$$\boldsymbol{E} = cB_0\boldsymbol{e}_x\cos(kz - \omega t).$$

(c)The Poyinting vector is

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H} = \boldsymbol{E}\times\frac{\boldsymbol{B}}{\mu_0} = cB_0\cos(kz - \omega t)\boldsymbol{e}_x\times\frac{1}{\mu_0}B_0\cos(kz - \omega t)\boldsymbol{e}_y = \frac{c}{\mu_0}B_0^2\cos^2(kz - \omega t)\boldsymbol{e}_z.$$

Making average over one period of time, we obtain

$$\langle\boldsymbol{S}\rangle = \frac{1}{T}\int_0^T \boldsymbol{S}\mathrm{d}t = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\boldsymbol{S}\mathrm{d}t = \frac{\omega}{2\pi}\frac{c}{\mu_0}B_0^2\int_0^{2\pi/\omega}\cos^2(kz - \omega t)\mathrm{d}t\,\boldsymbol{e}_z$$

$$= \frac{\omega}{2\pi}\frac{c}{\mu_0}B_0^2\int_0^{2\pi/\omega}\frac{1 + \cos(2kz - 2\omega t)}{2}\mathrm{d}t\,\boldsymbol{e}_z = \frac{1}{2}\frac{c}{\mu_0}B_0^2\,\boldsymbol{e}_z.$$