Mid-term Exam

1. There are three charges, q, q and – q, which are all placed on x-axis. The coordinates for them are $x = a$, $x = 0$, and $x = -a$, respectively. For the far zone, $|x| >> a$, and to the order of magnitude of dipole or a / x, what is the electric potential ϕ and field $\boldsymbol{E}$ on the point x?

Solution1:

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3},$$

$$\boldsymbol{p} = 2qa\boldsymbol{e}_x, \quad \boldsymbol{r} = x\boldsymbol{e}_x,$$

$$\varphi(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{1}{4\pi\varepsilon_0}\frac{2qax}{|x|^3}.$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3} + \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left[3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - r^2\boldsymbol{p}\right],$$

$$\boldsymbol{E}(x) = \frac{q}{4\pi\varepsilon_0}\frac{x}{|x|^3}\boldsymbol{e}_x + \frac{qa}{\pi\varepsilon_0}\frac{1}{|x|^3}\boldsymbol{e}_x.$$

$$3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - r^2\boldsymbol{p} = 6qax\cdot x\boldsymbol{e}_x - x^2 2qa\boldsymbol{e}_x = 4qax^2\boldsymbol{e}_x$$

Solution2:

$$\varphi(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{1}{4\pi\varepsilon_0}\frac{q}{|x-a|} - \frac{1}{4\pi\varepsilon_0}\frac{q}{|x+a|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|}\left(\frac{1}{|1-a/x|} - \frac{1}{|1+a/x|}\right)$$

$$\approx \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|}\left(\left|1+\frac{a}{x}\right| - \left|1-\frac{a}{x}\right|\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|x|} + \frac{1}{2\pi\varepsilon_0}\frac{qax}{|x|^3}. \qquad (10$$

When $x > 0$,

$$\boldsymbol{E}(x) = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\boldsymbol{e}_x + \frac{q}{4\pi\varepsilon_0}\frac{1}{(x-a)^2}\boldsymbol{e}_x - \frac{q}{4\pi\varepsilon_0}\frac{1}{(x+a)^2}\boldsymbol{e}_x$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\boldsymbol{e}_x + \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\left[\frac{1}{(1-a/x)^2} - \frac{1}{(1+a/x)^2}\right]\boldsymbol{e}_x$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\boldsymbol{e}_x + \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\frac{4a}{x}\boldsymbol{e}_x$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\boldsymbol{e}_x + \frac{qa}{\pi\varepsilon_0}\frac{1}{x^3}\boldsymbol{e}_x = \frac{q}{4\pi\varepsilon_0}\frac{x}{|x|^3}\boldsymbol{e}_x + \frac{qa}{\pi\varepsilon_0}\frac{1}{|x|^3}\boldsymbol{e}_x.$$

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When $x < 0$,

$$\boldsymbol{E}(x) = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}(-\boldsymbol{e}_x) + \frac{q}{4\pi\varepsilon_0}\frac{1}{(x-a)^2}(-\boldsymbol{e}_x) - \frac{q}{4\pi\varepsilon_0}\frac{1}{(x+a)^2}(-\boldsymbol{e}_x)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}(-\boldsymbol{e}_x) + \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\left[\frac{1}{(1-a/x)^2} - \frac{1}{(1+a/x)^2}\right](-\boldsymbol{e}_x)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}(-\boldsymbol{e}_x) + \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\frac{4a}{x}(-\boldsymbol{e}_x)$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}\boldsymbol{e}_x - \frac{qa}{\pi\varepsilon_0}\frac{1}{x^3}\boldsymbol{e}_x = \frac{q}{4\pi\varepsilon_0}\frac{x}{|x|^3}\boldsymbol{e}_x + \frac{qa}{\pi\varepsilon_0}\frac{1}{|x|^3}\boldsymbol{e}_x..$$

2. A long cylinder of radius R carries a charge density, $\rho = k\,r$, where k is a constant and r is the distance from the axis. (1) Find the electric field $\boldsymbol{E}$ inside and outside the cylinder. (2) Now, let the cylinder be shielded coaxially with a metal shell of radius R' find the energy U of the electric field per unit length.

Solution: (1) Make a coaxial cylindrical Gaussian surface of radius r and height h. For $r<R$, we have

$$\varepsilon_0 2\pi rhE = \int_0^r kr2\pi rhdr = 2\pi kh\int_0^r r^2dr = \frac{2}{3}\pi khr^3,$$

$$E = \frac{1}{3\varepsilon_0}kr^2, \boldsymbol{E} = \frac{1}{3\varepsilon_0}kr^2\boldsymbol{e}_r.$$

For $r>R$, we have

$$\varepsilon_0 2\pi rhE = \int_0^R kr2\pi rhdr = 2\pi kh\int_0^R r^2dr = \frac{2}{3}\pi khR^3,$$

$$E = \frac{1}{3\varepsilon_0 r}kR^3, \boldsymbol{E} = \frac{1}{3\varepsilon_0 r}kR^3\boldsymbol{e}_r.$$

(2)

$$u = \frac{\varepsilon_0}{2} E^2,$$

$$U = \int_0^{R'} u \cdot 2\pi r dr \cdot 1 = \frac{\varepsilon_0}{2} \int_0^{R'} E^2 2\pi r dr = \frac{\varepsilon_0}{2} \left(\int_0^{R} E^2 2\pi r dr + \int_R^{R'} E^2 2\pi r dr \right)$$

$$= \frac{\varepsilon_0}{2} \left(\int_0^{R} \left(\frac{kr^2}{3\varepsilon_0} \right)^2 2\pi r dr + \int_R^{R'} \left(\frac{kR^3}{3\varepsilon_0 r} \right)^2 2\pi r dr \right)$$

$$= \frac{k^2 \pi}{9\varepsilon_0} \left(\int_0^{R} r^5 dr + R^6 \int_R^{R'} \frac{dr}{r} \right) = \frac{k^2 \pi R^6}{9\varepsilon_0} \left(\frac{1}{6} + \ln \frac{R'}{R} \right)$$

3. A parallel-plate capacitor has area A and separation d. The upper half of the capacitor is filled with an insulator of relative dielectric constant ε_1, and the lower half with another insulator of relative dielectric constant ε_2. (1) Find the capacitance C of the capacitor. (2) Find the areal charge density σ of the interface between the two insulators if the capacitor is charged to a voltage V.

Solution: (1) Let σ_0 be the areal charge density on the plate. Making cylindrical Gaussian surface of base area S with one base inside the conductor plate and one base inside the dielectrics, the Gauss' law (of the electric displacement) can be written as $DS = \sigma_0 S$.

Thus, the electric displacement is $D = \sigma_0$ and the electric field is

$$E = \begin{cases} E_1 = \dfrac{\sigma_0}{\varepsilon_1 \varepsilon_0} & \text{for upper half of the capacitor,} \\ E_2 = \dfrac{\sigma_0}{\varepsilon_2 \varepsilon_0} & \text{for lower half of the capacitor.} \end{cases}$$

Then, the potential difference (voltage) across the two plates of the capacitor is

$$V = \frac{\sigma_0}{\varepsilon_1\varepsilon_0}\frac{d}{2} + \frac{\sigma_0}{\varepsilon_2\varepsilon_0}\frac{d}{2} = \frac{Qd}{2\varepsilon_1\varepsilon_0 A} + \frac{Qd}{2\varepsilon_2\varepsilon_0 A}$$

$$= \frac{d}{2\varepsilon_0 A}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)Q$$

Therefore, the capacitance is

$$C = \frac{Q}{V} = \left[\frac{d}{2\varepsilon_0 A}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\right]^{-1} = \frac{2\varepsilon_0\varepsilon_1\varepsilon_2 A}{d\left(\varepsilon_1 + \varepsilon_2\right)}$$

(2) Making a pillbox like Gaussian surface with two bases parallel to the interface and one base in each dielectrics, the Gauss's law (for the electric field) reads

$$\varepsilon_0\left(-E_1 S + E_2 S\right) = \sigma S$$

where σ is the polarization areal charge density (surface bound charge density). Using the result of the electric field in (1), we obtain

$$\sigma = \varepsilon_0\left(E_2 - E_1\right) = \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\sigma_0$$

$$= \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{Q}{A} = \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{CV}{A}$$

$$= \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{2\varepsilon_0\varepsilon_1\varepsilon_2 V}{d\left(\varepsilon_1 + \varepsilon_2\right)} = \frac{2\varepsilon_0\left(\varepsilon_1 - \varepsilon_2\right)V}{\left(\varepsilon_1 + \varepsilon_2\right)d}.$$

4. Find the self-inductance L of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), which carries a total of N turns.

Solution:Making a loop that is concentric with toroidal coil and using the Ampere's law, we have

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = 2\pi r B = \mu_0 I = \mu_0 N i.$$

Thus, we obtain

$$B = \frac{\mu_0 N i.}{2\pi r}.$$

The magnetic flux that passes through a turn of the coil is

$$\phi_m = \int_a^b B h dr = \int_a^b \frac{\mu_0 N i}{2\pi r} h \mathrm{d}r = \frac{\mu_0}{2\pi} N i h \ln\left(\frac{b}{a}\right).$$

The total flux is

$$\Phi_m = N\phi_m = \frac{\mu_0}{2\pi} N^2 i h \ln\left(\frac{b}{a}\right).$$

Finally, we obtain that the self-inductance is given by

$$L = {\Phi_m}/{i} = \frac{\mu_0}{2\pi} N^2 h \ln\left(\frac{b}{a}\right).$$

5. A soap bubble of radius r is in equilibrium with the outer air of pressure p. Now, it is slowly given a charge q, the radius of the bubble will increase gradually to R, due to the mutual repulsion of the surface charge. Regard the air as an ideal gas, and neglect the effect of the tension of the surface of the bubble. (1) Find the pressure p_{field} of the electric field upon the bubble, the dielectric constant of the outer air being ε. (2) Show that $q^2 = 32\pi^2 \varepsilon p R (R^3 - r^3)$.

Solution: (1) The force acting on a area element dS on the surface of the bubble due to the electric charge is

$$dF = \frac{1}{2} E\sigma dS$$

where

$$E = \frac{1}{4\pi\varepsilon} \frac{q}{R^2}$$

is the electric field in the vicinity of the surface and

$$\sigma = \frac{q}{4\pi R^2}$$

is the surface electric charge density. Thus

$$p_{field} = \frac{q^2}{32\pi^2 \varepsilon} \frac{1}{R^4}.$$

(2) As the bubble expands, the pressure of the gas inside reduces to

$$p' = \frac{V}{V'} p = \frac{r^3}{R^3} p.$$

The equilibrium requires

$$p' + p_{\text{field}} = p,$$

or

$$p - p' = p_{\text{field}}, \qquad (1 - \frac{r^3}{R^3})\,p = \frac{q^2}{32\pi^2\varepsilon}\frac{1}{R^4}.$$

Finally, we obtain

$$q^2 = 32\pi^2\varepsilon R^4(1 - \frac{r^3}{R^3})\,p = 32\pi^2\varepsilon R(R^3 - r^3)\,p.$$

6. There are two electromagnetic travelling waves overlapped in free space, their electric fields are $\boldsymbol{E}_1 = C\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\,\boldsymbol{e}_x$ and $\boldsymbol{E}_2 = D\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\,\boldsymbol{e}_x + D\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\,\boldsymbol{e}_y$, where $C > 0$ and $D > 0$ are constant, and ω and $\boldsymbol{k} = k\,\boldsymbol{e}_z$ are the frequency and wave vector ($\omega = c\,k$ where c is the speed of light), respectively. Here, x, y and z are Cartesian coordinates, and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ the corresponding unit vectors, t denotes the time, and $\boldsymbol{r} = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z$. (1) Find the total magnetic induction $\boldsymbol{B}$. (2) Find the energy density u of the electromagnetic field and its time average $< u >$ over one period. (3) Find the Poynting vector $\boldsymbol{S}$ and its time average $< \boldsymbol{S} >$ over one period.

Solution: (1) The total electric field is

$$
\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_1 + \boldsymbol{E}_2 \\
&= (C + D)\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\boldsymbol{e}_x + D\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\boldsymbol{e}_y .
\end{aligned}
$$

By using the Maxwell's equations (Faraday's law)

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

we have

$$k\boldsymbol{e}_z \times [-(C+D)\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x - D\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y] = -\frac{\partial \boldsymbol{B}}{\partial t}$$

or

$$k[(C+D)\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y + D\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x] = \frac{\partial \boldsymbol{B}}{\partial t}.$$

Finally

$$\boldsymbol{B} = \frac{k}{\omega}[(C+D)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y + D\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x]$$

$$= \frac{1}{c}[(C+D)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y + D\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x] \ .$$

(2)

$$u=\frac{1}{2}\varepsilon_0 \boldsymbol{E}\cdot\boldsymbol{E}+\frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}=\ \varepsilon_0\boldsymbol{E}\cdot\boldsymbol{E}$$

$$=\varepsilon_0\left[(C+D)^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)+D^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\right].$$

$$\langle u\rangle=\frac{1}{T}\int_0^T u\mathrm{d}t=\varepsilon_0\left[(C+D)^2+D^2\right]\left(\frac{1}{T}\int_0^T\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\mathrm{d}t\right)$$

$$=\frac{\varepsilon_0}{2}\left[(C+D)^2+D^2\right].$$

$$\boldsymbol{E}=(C+D)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x+D\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y.$$

$$\boldsymbol{B}=\frac{1}{c}[(C+D)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y+D\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x]\ .$$

(3)

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$$

$$= \frac{1}{\mu_0 c}\left[(C+D)^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\left(\boldsymbol{e}_x\times\boldsymbol{e}_y\right)+D^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\left(\boldsymbol{e}_y\times\boldsymbol{e}_x\right)\right]$$

$$= \varepsilon_0 c\left[(C+D)^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)-D^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\right]\boldsymbol{e}_z.$$

$$\left\langle \boldsymbol{S}\right\rangle = \frac{\varepsilon_0 c}{2}\left[(C+D)^2-D^2\right]\boldsymbol{e}_z.$$

$$\boldsymbol{E}=(C+D)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x+D\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y.$$

$$\boldsymbol{B}=\frac{1}{c}[(C+D)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_y+D\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\boldsymbol{e}_x]\ .$$