

Sample Mid-term 2

1. The electric charge density of a spherically symmetric charge distribution is

$$\rho(r) = \begin{cases} \rho_0 \left(1 - \frac{r^2}{a^2}\right), & r < a, \\ 0, & r > a, \end{cases} \quad (a > 0).$$

(a) Calculate the total charge

(b) Find the electric field \mathbf{E} and the electric potential $V(r)$ in the region $r > a$.

(c) Find the electric field \mathbf{E} and the electric potential $V(r)$ in the region $r < a$.

Solution: (a) $Q = \int_0^a \rho_0 \left(1 - \frac{r^2}{a^2}\right) 4\pi r^2 dr$

$$= 4\pi\rho_0 \left(\frac{1}{3} r^3 - \frac{r^5}{5a^2} \right)_0^a = \frac{8\pi}{15} \rho_0 a^3.$$

(b) When $r > a$, using the Gauss's law, we have

$$\epsilon_0 4\pi r^2 E = Q,$$

or

$$E = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{2\rho_0 a^3}{15\epsilon_0 r^2}.$$

The electric potential is

$$V(r) = -\int_{\infty}^r E(r) dr = \frac{2\rho_0 a^3}{15\epsilon_0 r}.$$

$$Q = \int_0^a \int_0^\pi \int_0^{2\pi} \rho(r) r^2 \sin \theta dr d\theta d\phi$$

$$= \int_0^a \rho(r) 4\pi r^2 dr$$


$$\int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\phi = 4\pi$$

(c) When $r < a$, using the Gauss's law, we have

$$\begin{aligned}\epsilon_0 4\pi r^2 E &= \int_0^r \rho_0 \left(1 - \frac{r^2}{a^2}\right) 4\pi r^2 dr \\ &= 4\pi \rho_0 \left(\frac{1}{3} r^3 - \frac{r^5}{5a^2} \right),\end{aligned}$$

or

$$E = \frac{\rho_0}{\epsilon_0} \left(\frac{1}{3} r - \frac{r^3}{5a^2} \right).$$

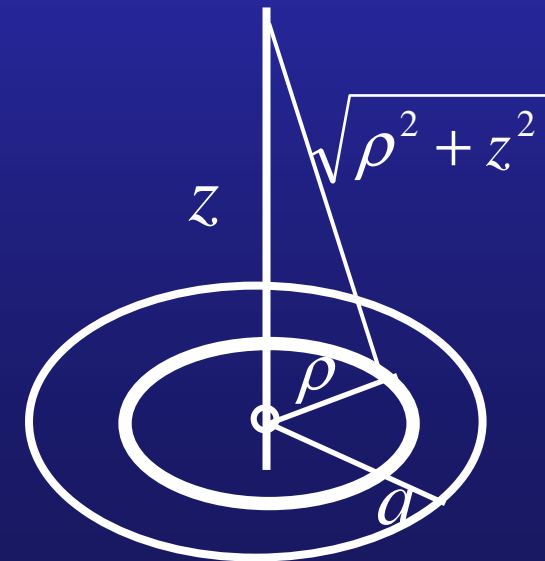
The electric potential is

$$\begin{aligned} V(r) &= -\int_{\infty}^r E(r) dr \\ &= -\int_{\infty}^a E(r) dr - \int_a^r E(r) dr \\ &= \frac{2\rho_0}{15\varepsilon_0} a^2 - \frac{\rho_0}{\varepsilon_0} \left(\frac{r^2}{6} - \frac{r^4}{20a^2} - \frac{7}{60} a^2 \right) \\ &= \frac{\rho_0}{4\varepsilon_0} a^2 - \frac{\rho_0}{\varepsilon_0} \left(\frac{r^2}{6} - \frac{r^4}{20a^2} \right) \end{aligned}$$

2. A circular disk of radius a has the surface charge density σ .

(a) Compute the electric potential at a point on the axis of symmetry and at distance z from the center of the disk and then the electric field there.

(b) Find the discontinuity of the electric field at the center of the disk. Reobtain the result by using the Gauss's law of the electric field.

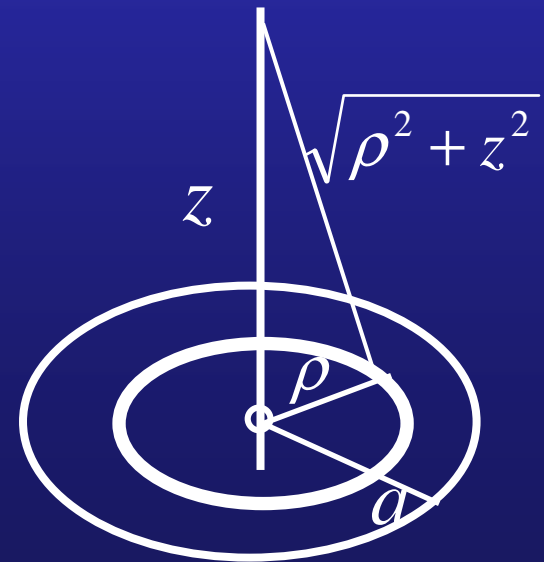


Solution: (a) The electric potential is

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_0^a \frac{\sigma \cdot 2\pi\rho d\rho}{\sqrt{\rho^2 + z^2}} = \frac{\sigma}{4\epsilon_0} \int_0^a \frac{d\rho^2}{\sqrt{\rho^2 + z^2}}$$

$$= \frac{\sigma}{4\epsilon_0} \int_0^a \frac{d(\rho^2 + z^2)}{\sqrt{\rho^2 + z^2}} = \frac{\sigma}{4\epsilon_0} \int_{z^2}^{a^2 + z^2} \frac{dt}{\sqrt{t}}$$

$$= \frac{\sigma}{2\epsilon_0} \left(\sqrt{a^2 + z^2} - |z| \right)$$



The electric field on the axis is

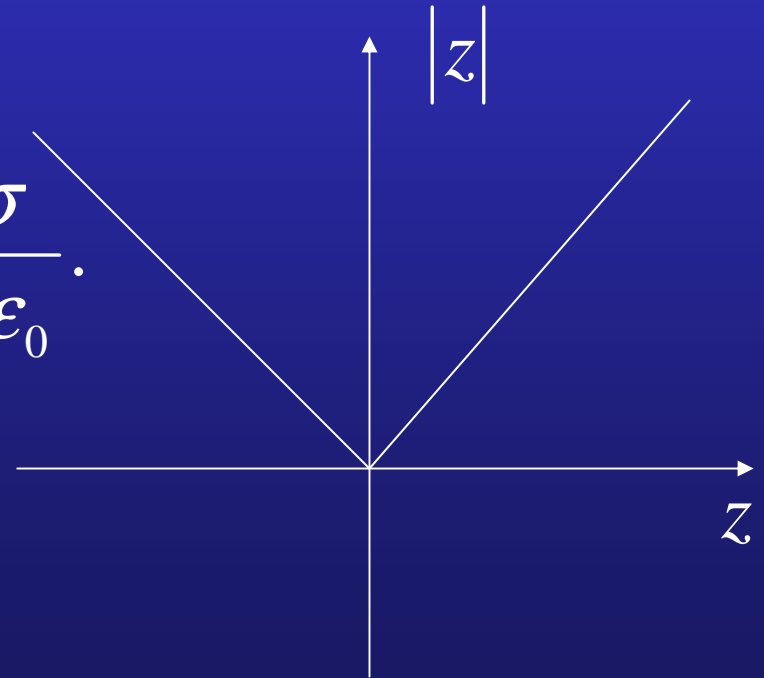
$$E_z(z) = -\frac{\partial V(z)}{\partial z} = \frac{\sigma}{2\epsilon_0} \left(\text{sgn}(z) - \frac{z}{\sqrt{a^2 + z^2}} \right)$$

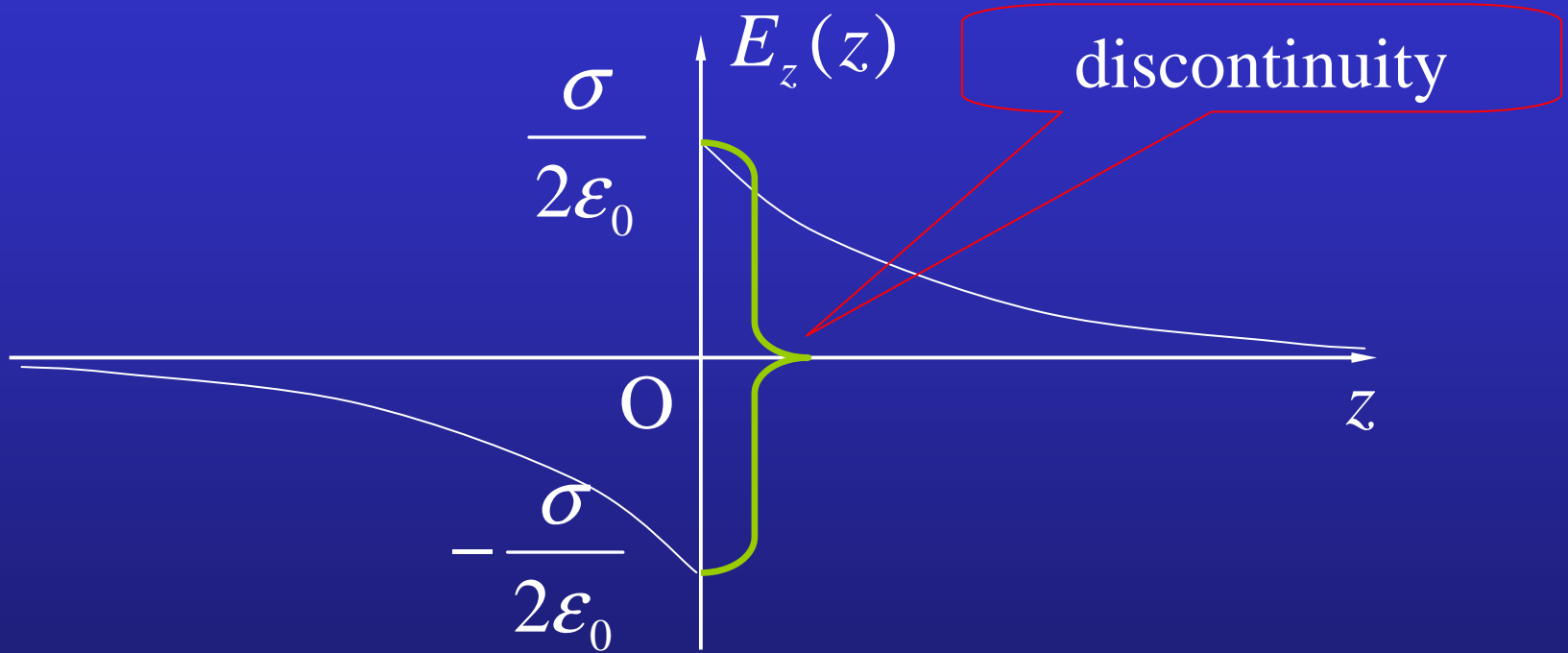
(b) From the result of (a) we have

$$E_z(0^+) = \frac{\sigma}{2\epsilon_0}, \quad E_z(0^-) = -\frac{\sigma}{2\epsilon_0}.$$

Thus, the discontinuity is

$$E_z(0^+) - E_z(0^-) = \frac{\sigma}{\epsilon_0}.$$





Making a pillbox-like Gaussian surface around the center of the disk, we have

$$2\epsilon_0 ES = \sigma S,$$

where S is the area of the two bases of the Gaussian surface that are parallel to the disk. Finally, we obtain that

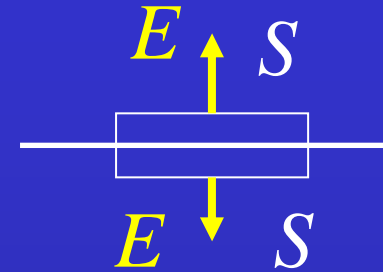
$$E = \frac{\sigma}{2\epsilon_0}$$

or

$$E_z(0^+) = \frac{\sigma}{2\epsilon_0}, \quad E_z(0^-) = -\frac{\sigma}{2\epsilon_0}.$$

Therefore,

$$E_z(0^+) - E_z(0^-) = \frac{\sigma}{\epsilon_0}.$$



3. A steady electric current I in the z -direction flows uniformly in the region between two cylindrical surfaces $x^2 + y^2 = a^2$ and $(x - a/2)^2 + y^2 = b^2$, where $a > 0$, $b > 0$ and $b < a/2$.

(a) Find the magnitude and direction of the magnetic field \mathbf{B} at every point P on the x -axis.

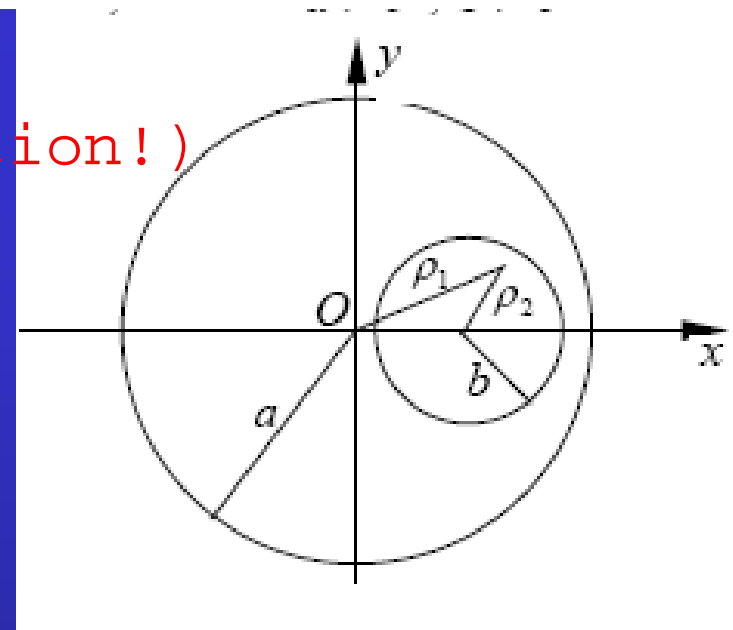
(b) Show that the magnetic field \mathbf{B} is uniform in the region $(x - a/2)^2 + y^2 < b^2$ and the field is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{a}{a^2 - b^2} \mathbf{e}_y .$$

(Hint: A vector in the x - y plane that is perpendicular to the displacement $\boldsymbol{\rho}$ in the x - y plane and of the magnitude $\rho = |\boldsymbol{\rho}|$ can be expressed as $\pm \mathbf{e}_z \times \boldsymbol{\rho}$.)

Solution : (a) (incomplete solution!)

$$j = \frac{I}{S} = \frac{I}{\pi(a^2 - b^2)}$$

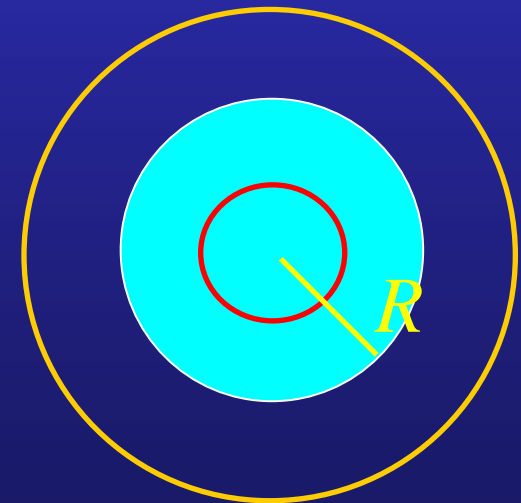


$$\rho < R, \quad 2\pi\rho B = \mu_0\pi\rho^2 j$$

$$\underline{B = \frac{\mu_0 j}{2} \rho}$$

$$\rho > R, \quad 2\pi\rho B = \mu_0\pi R^2 j$$

$$\underline{B = \frac{\mu_0 j}{2} \frac{R^2}{\rho}}$$



(b) In the region $(x - a/2)^2 + y^2 < b^2$

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$$

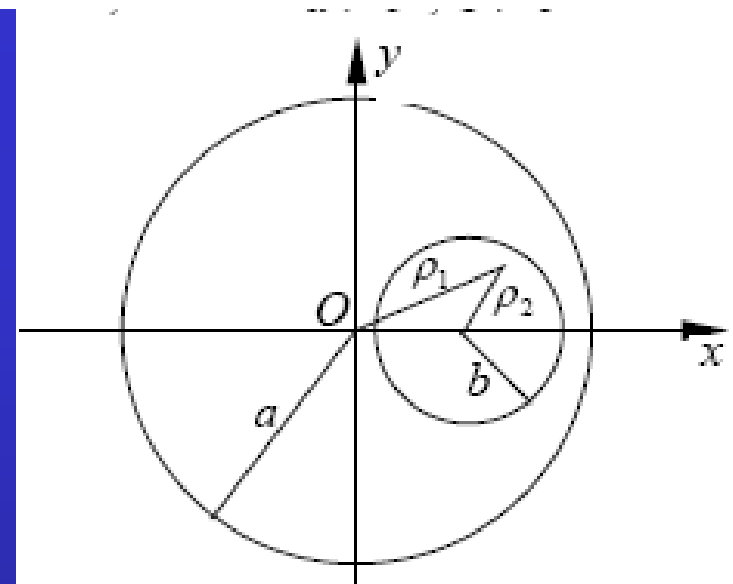
$$\mathbf{B}_1 = \frac{\mu_0 j}{2} \rho_1, \quad \mathbf{B}_1 = \frac{\mu_0 j}{2} \mathbf{e}_z \times \rho_1,$$

$$\mathbf{B}_2 = \frac{\mu_0 j}{2} \rho_2, \quad \mathbf{B}_2 = -\frac{\mu_0 j}{2} \mathbf{e}_z \times \rho_2,$$

$$\rho_1 = \frac{1}{2} a \mathbf{e}_x + \rho_2,$$

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \frac{\mu_0 j}{2} \mathbf{e}_z \times (\rho_1 - \rho_2)$$

$$= \frac{\mu_0 j}{2} \mathbf{e}_z \times \left(\frac{a}{2} \mathbf{e}_x \right) = \frac{\mu_0 j}{4\pi} \frac{a}{a^2 - b^2} \mathbf{e}_y$$



\vec{B} $\vec{\rho}$ \vec{r}

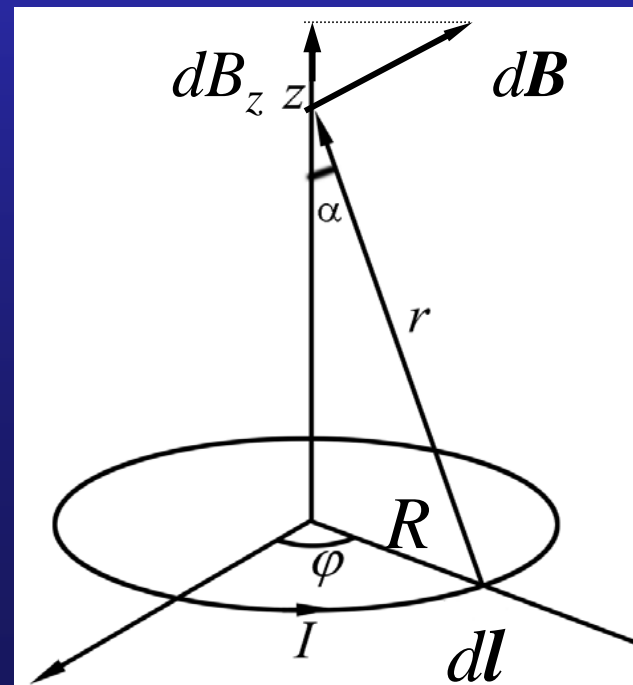
\vec{e}_x \vec{e}_y \vec{e}_z

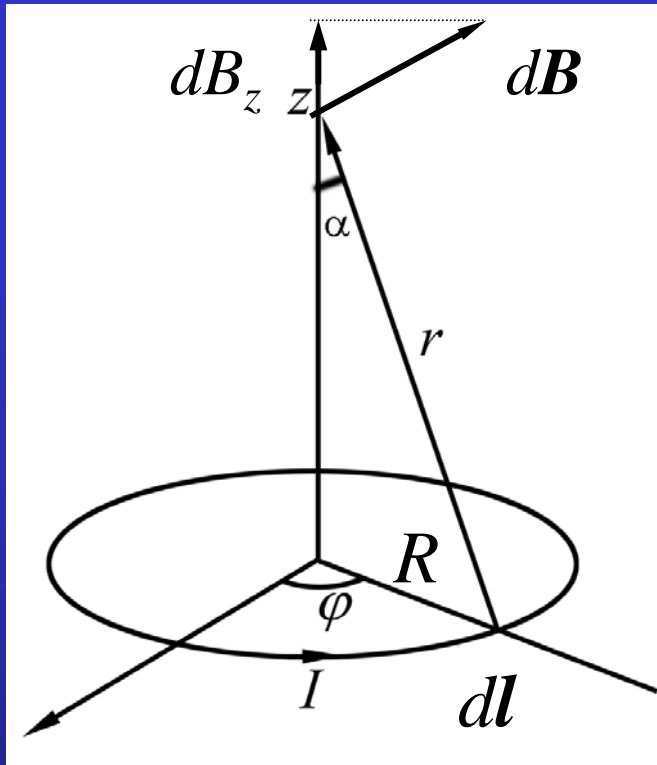
$\vec{\rho}$ \vec{r}

4. As shown in the accompanying figure, a loop of radius R carries an electric current I .

(a) Find the magnetic field at point P that is on the symmetric axis of the loop and at distance z from the center of the loop.

(b) If a magnetic dipole with magnetic moment $\mathbf{m} = m_0 \mathbf{e}_z$ is placed at point P , what is the magnitude and direction of the force acted on the dipole?





Solution: (a) We see in the figure that r and dl are perpendicular. So, according to Biot-Savart law,

$$dB = \frac{\mu_0}{4\pi} \frac{I dl}{r^2}.$$

The component along the axis is

$$dB_z = \frac{\mu_0}{4\pi} \frac{I dl}{r^2} \sin \alpha = \frac{\mu_0 I}{4\pi} \frac{R}{r^3} dl.$$

Integration around the loop gives

$$B_z = \frac{\mu_0 I}{4\pi} \frac{R}{r^3} (2\pi R) = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}}.$$

(b) The potential energy of the magnetic dipole is

$$U(z) = -\mathbf{m} \cdot \mathbf{B} = -m_0 B_z.$$

Thus, the force acted on the dipole is

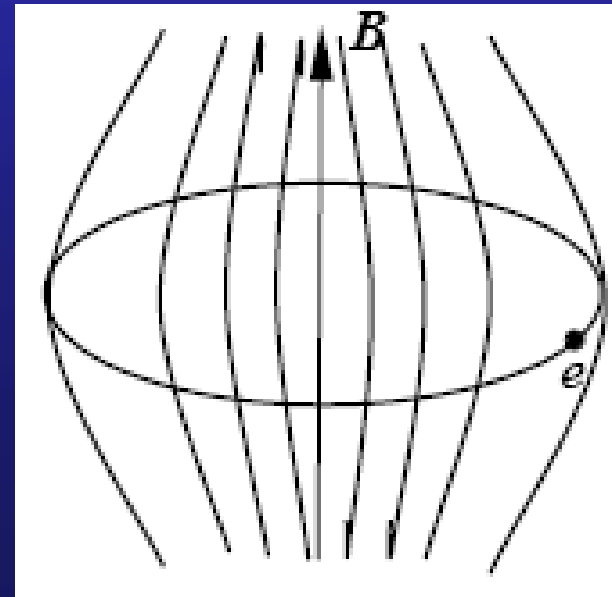
$$F_z = -\frac{\partial U(z)}{\partial z} = m_0 \frac{\partial B_z}{\partial z} = -\frac{3\mu_0 m_0 I}{2} \frac{R^2 z}{(z^2 + R^2)^{5/2}}$$

$$(\mathbf{F} = -\nabla U = \nabla(\mathbf{m} \cdot \mathbf{B}) = \mathbf{m} \cdot \nabla \mathbf{B})$$

5. The betatron is a cyclic induction electron accelerator. In a betatron, the electron is accelerated by the circular electric field induced by the changing magnetic field that is axially symmetric. Simultaneously, the magnetic field also provides the centripetal force for the electron to move in a circular orbit. The poles of the magnet are tapered to cause the axially symmetric magnetic field to weaken with increasing radius. Show that if the average magnetic field inside the orbit

$$\bar{B} = \frac{1}{\pi R^2} \int_0^R B(\rho) 2\pi\rho d\rho$$

is always twice as strong as the magnetic field on the orbit $B(R)$, i.e., $\bar{B}=2B(R)$, the electron can be accelerated stably on the orbit with fixed radius R .



Solution :

$$\varepsilon = 2\pi R E = -\frac{\partial \Phi_B}{\partial t} = -\pi R^2 \frac{\partial \bar{B}}{\partial t}$$

$$E = -\frac{R}{2} \frac{\partial \bar{B}}{\partial t}$$

$$F_T = -eE = \frac{eR}{2} \frac{\partial \bar{B}}{\partial t},$$

$$F_N = evB(R).$$

$$m \frac{v^2}{R} = p \frac{v}{R} = evB(R).$$



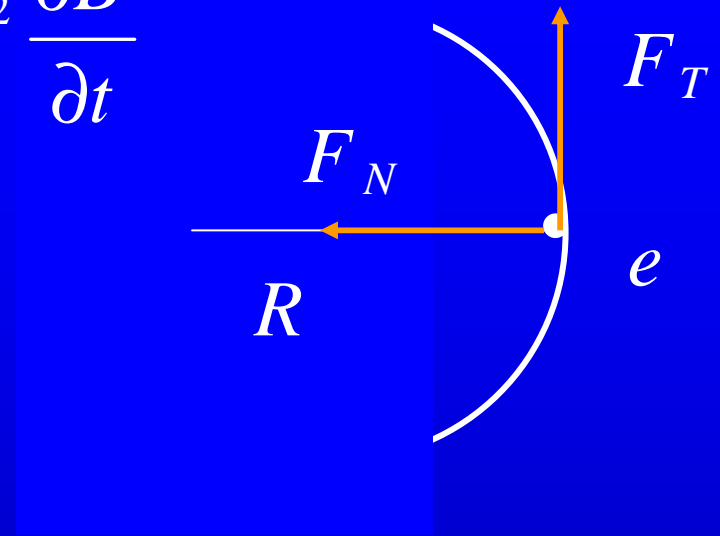
$$p = eRB(R).$$

$$\frac{dp}{dt} = eR \frac{\partial B(R)}{\partial t},$$

$$2B(R) = \bar{B}$$

$$\frac{dp}{dt} = F_T = \frac{eR}{2} \frac{\partial \bar{B}}{\partial t},$$

condition for stable
cyclotron motion



6. A non-uniform time-varying electric field given by

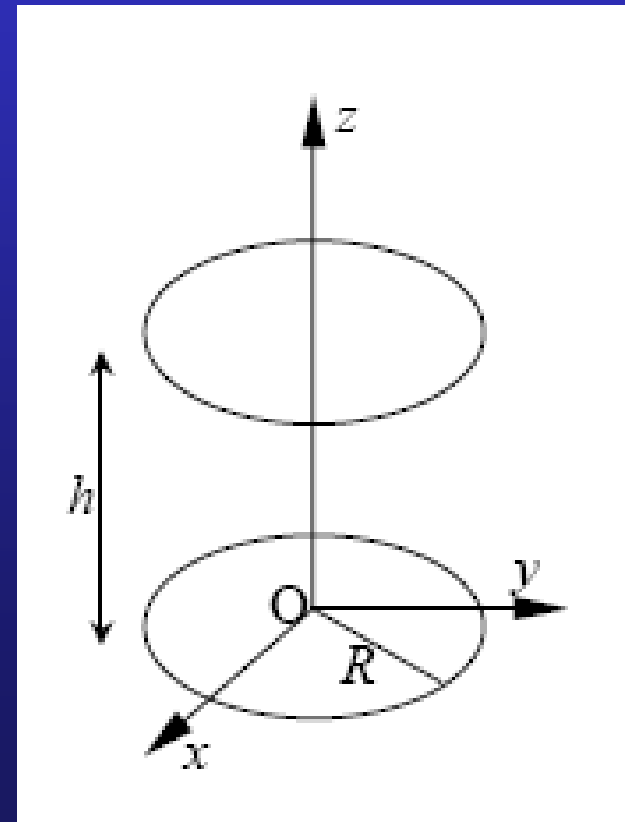
$$\mathbf{E} = E_0 \left(1 - \frac{\rho}{R} \right) \sin(\omega t) \mathbf{e}_z$$

is distributed in the cylindrical region of radius R and height h as shown in the figure.

(a) Find the displacement current density j_d in the cylindrical region ($\rho < R$).

(b) Find the magnetic field \mathbf{B} induced in the same region.

(c) Indicate on the figure the direction of \mathbf{B} at radial distance ρ from the axis at $t = 0$.



Solution: (a)

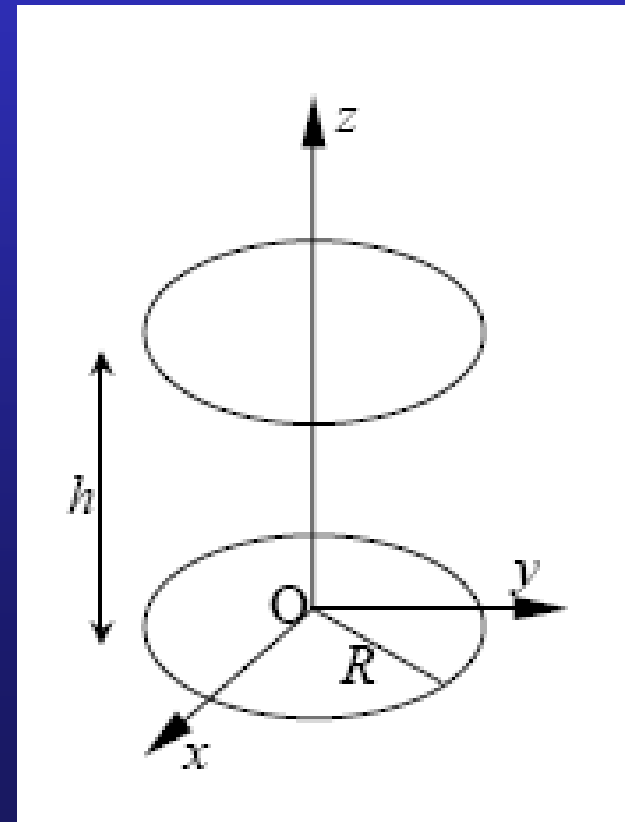
$$\mathbf{j}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \varepsilon_0 E_0 \omega \left(1 - \frac{\rho}{R}\right) \cos(\omega t) \mathbf{e}_z$$

(b)

$$I_d = \int_0^{\rho} j_d(\rho) 2\pi\rho d\rho$$

$$= 2\pi\varepsilon_0 E_0 \omega \cos(\omega t) \int_0^{\rho} \left(1 - \frac{\rho}{R}\right) \rho d\rho$$

$$= 2\pi\varepsilon_0 E_0 \omega \cos(\omega t) \left(\frac{\rho^2}{2} - \frac{\rho^3}{3R} \right)$$



By using Ampere-Maxwell law, we have

$$2\pi\rho B(\rho) = \mu_0 I_d.$$

Therefore,

$$B(\rho) = \varepsilon_0 \mu_0 E_0 \omega \cos(\omega t) \left(\frac{\rho}{2} - \frac{\rho^2}{3R} \right)$$
$$= \frac{E_0}{c^2} \omega \cos(\omega t) \left(\frac{\rho}{2} - \frac{\rho^2}{3R} \right).$$

(c) As shown in the figure.

