1	2	3	4	5	6	Sum
/20	/20	/20	/20	/20	/20	/100

Name ______________________ Nov. 6, 2003 Thursday

Department _________________

SID _______________________

University Physics

Midterm Examination

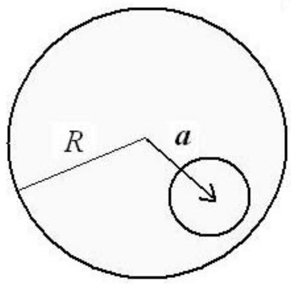
School of Intensive Instruction in Sciences and Arts, Nanjing University

Physical Constants

Electron charge	$e = 1.60 \times 10^{-19}$ C
Dielectric constant of free space	$\varepsilon_0 = 8.85 \times 10^{-12}$ F/m
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ N/A^2
Vacuum speed of light	$c = 3.00 \times 10^8$ m/s
Electron mass	$m_e = 9.11 \times 10^{-31}$ kg
Proton mass	$m_n = 1.67 \times 10^{-27}$ kg
Electron volt (eV)	1eV=1.60×10^{-19} J

Select five out of following six problems.

1 (20pts). A nonconducting sphere of radius R carries a uniform charge per unit volume ρ. Let $\boldsymbol{r}$ be the vector from the center of the sphere to a point P.



(a) Show that the electric field at P is given by $\boldsymbol{E} = \rho\ \boldsymbol{r}/3\varepsilon_0$ when the point P is within the sphere.

(b) Show that the electric field at P is given by $\boldsymbol{E} = \rho R^3 \boldsymbol{r}/3\varepsilon_0 r^3$ when the point P is outside the sphere.

(c) A spherical cavity is created in the sphere, as shown in the accompanying figure. In this case, show that the electric field at all points within the cavity is $\boldsymbol{E} = \rho\ \boldsymbol{a}/3\varepsilon_0$ (uniform field), where $\boldsymbol{a}$ is the vector connecting the center of the charged sphere and the center of the cavity.

Solution: (a) Making a spherical surface of radius $r < R$ that is concentric to the charged sphere as the Gaussian surface, we have

$$\varepsilon_0 4\pi r^2 \cdot E = \frac{4}{3}\pi r^3 \rho \quad \text{or} \quad E = \rho\ r/3\varepsilon_0.$$

It can be written in the vector form as $\boldsymbol{E} = \rho\ \boldsymbol{r}/3\varepsilon_0$.

(b) Making a spherical surface of radius $r > R$ that is concentric to the charged sphere as the Gaussian surface, we have

$$\varepsilon_0 4\pi r^2 \cdot E = \frac{4}{3}\pi R^3 \rho \quad \text{or} \quad E = \rho R^3/3\varepsilon_0 r^2.$$

It can be written in the vector form as $\boldsymbol{E} = \rho R^3 \boldsymbol{r}/3\varepsilon_0 r^3$.

(c) Creating a spherical cavity in the sphere is equivalent to impose a sphere with equal and opposite charge density in the nonconducting sphere. The field produced by the imposed sphere (inside the sphere) is then $\boldsymbol{E}' = -\rho\ \boldsymbol{r}'/3\varepsilon_0$, where $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{a}$. By using the principle of superposition, the resultant electric field inside the cavity is

$$\boldsymbol{E} = \rho\ \boldsymbol{r}/3\varepsilon_0 + \boldsymbol{E}' = \rho\ \boldsymbol{r}/3\varepsilon_0 - \rho\ \boldsymbol{r}'/3\varepsilon = \rho\ \boldsymbol{a}/3\varepsilon_0.$$

2.(20pts). A uniformly charged rod of length L with linear charge density λ is placed on the z-axis from $z = -L/2$ to $z = L/2$.

(a) Obtain the electrical potential $V(x, y)$ for any point on the x-y plane.

(b) What would you expect the potential to be for $\sqrt{x^2 + y^2} >> L$?

(c) What is the electric field in the x-y plane in the limit that $\sqrt{x^2 + y^2} << L$?

Solution: (a) The electrical potential produced by the charge in the element dz at point (x, y) on the x-y plane can be written as

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dz}{\sqrt{x^2 + y^2 + z^2}}.$$

Thus, the resultant potential is

$$V(x, y) = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{\lambda dz}{\sqrt{x^2 + y^2 + z^2}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln(z + \sqrt{x^2 + y^2 + z^2})\right]_{z=-L/2}^{z=L/2}$$
$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{(\sqrt{4x^2 + 4y^2 + L^2} + L)}{(\sqrt{4x^2 + 4y^2 + L^2} - L)}.$$

(b) When $\sqrt{x^2 + y^2} >> L$

$$V(x, y) = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{(\sqrt{4x^2 + 4y^2 + L^2} + L)}{(\sqrt{4x^2 + 4y^2 + L^2} - L)} = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{(1 + L/\sqrt{4x^2 + 4y^2 + L^2})}{(1 - L/\sqrt{4x^2 + 4y^2 + L^2})}$$
$$\approx \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{\sqrt{4x^2 + 4y^2 + L^2}} \approx \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{\sqrt{x^2 + y^2}}.$$

That is the potential of a point charge at the origin.

(c) Noting that the z-component of electric field produced by the charge element at z cancels with the z-component of electric field produced by the charge element at $-z$, the nonzero component of the electric field is within the x-y plane. We therefore have

$$dE_\rho = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dz}{\rho^2 + z^2}\frac{\rho}{\sqrt{\rho^2 + z^2}} \quad \text{where} \quad \rho = \sqrt{x^2 + y^2}.$$

The total field is

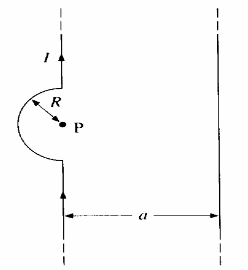
$$E_\rho = \int dE_\rho == \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{\lambda\rho dz}{(\rho^2 + z^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{\rho\sqrt{\rho^2 + L^2/4}}.$$

In the case of $\rho << L$, we have

$$E_\rho \approx \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{\rho\sqrt{L^2/4}} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{\rho}.$$

That is the field of a uniformly charged infinite rectilinear wire.

3. (20pts). Two long wires, one of which has a semicircular bend of radius R, are positioned as shown in the accompanying figure. Both wires carry a current I.



(a) How far apart must they be so that the resultant magnetic field at the point P is zero?

(b) Does the current in the straight wire flow up or down?

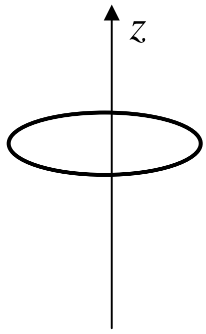
Solution: The field produced by the current on the semicircular bend at the point P is $B_b = \mu_0 I/4R$. The direction of the field is into the paper. Assuming the separation between to wire is d, the field produced by the current on the straight wire at P is $B_s = \mu_0 I/2\pi d$. Because the two fields cancel each other, we have

$$\mu_0 I/4R = \mu_0 I/2\pi d \quad \text{or} \quad d = \frac{2}{\pi}R.$$

(b) It flows upward.

4.(20pts). A circular wire loop of radius r falls under the influence of gravity and in the presence of a magnetic field given by

$$\boldsymbol{B} = B_0(1+az)\boldsymbol{e}_z - \frac{aB_0\rho}{2}\boldsymbol{e}_\rho$$



in the cylindrical coordinates, here a is a constant. The loop remains horizontal while falling. The wire has the mass m and the resistance of the loop is R.

(a) Find the induced emf in the loop when the loop falls down with the velocity v.

(b) Assuming that $a > 0$, determine the direction of the induced current. Explain.

(c) Show that the terminal velocity of the loop is $v_t = \dfrac{mgR}{a^2 B_0^2 \pi^2 r^4}$.

Solution: (a) The emf is

$$E = -\frac{d\Phi_m}{dt} = -\frac{d(B_z \pi r^2)}{dt} = -B_0 \pi r^2 a \frac{dz}{dt} = B_0 \pi r^2 av .$$

(b) When $a > 0$, as the loop falls down, the flux through it decreases. Then the induced current must produce a field that is in the same direction of the applied field. Thus, the field produced by the current is upward and the induced current on the loop flows counter-clockwise.

The magnitude of the induced current is

$$I = \frac{E}{R} = \frac{B_0 \pi r^2 av}{R}.$$

(c) The magnetic force acted on a current element of the loop is

$$d\boldsymbol{F} = Idl\,\boldsymbol{e}_\phi \times \boldsymbol{B} = Idl\,\boldsymbol{e}_\phi \times \left[B_0(1+az)\boldsymbol{e}_z - \frac{aB_0 r}{2}\boldsymbol{e}_\rho \right]$$

$$= IB_0(1+az)dl\boldsymbol{e}_\rho + \frac{IaB_0 r}{2} dl\boldsymbol{e}_z .$$

The first term cancels after the integration. Thus the resultant force is

$$F = \frac{IaB_0 r}{2} \cdot 2\pi r = \frac{a^2 B_0^2 \pi^2 r^4 v}{R}.$$

The terminal velocity is reached when the gravity and the magnetic force are balanced. Therefore

$$\frac{a^2 B_0^2 \pi^2 r^4 v_t}{R} = mg \quad \text{or} \quad v_t = \frac{mgR}{a^2 B_0^2 \pi^2 r^4} .$$

5. (20pts). A CD (thin insulating disk with a hole in the middle, inner radius a, out radius b) spins counter-clockwise in the x-y plane with angular velocity ω. The mass of the CD disk is m. A total charge Q is distributed uniformly on the surface of the disk.

(a) Find the magnetic moment of the spinning disk.

(b) Suppose a uniform magnetic field B is applied in the y-direction. What is the angular velocity with which the disk will precess?

Solution: (a) The area charge density is

$$\sigma = \frac{Q}{\pi(b^2 - a^2)} .$$

The magnetic moment is

$$\mu = \int_a^b \sigma\omega\rho d\rho \cdot \pi\rho^2 = \frac{\pi\sigma\omega}{4}\left(b^4 - a^4\right) = \frac{\pi\omega}{4}\left(b^4 - a^4\right)\cdot\frac{Q}{\pi(b^2-a^2)} = \frac{Q\omega}{4}\left(b^2 + a^2\right) \text{or} \quad \boldsymbol{\mu} = \frac{Q\omega}{4}\left(b^2 + a^2\right)\boldsymbol{e}_z$$

(b) The moment of inertia of the disk is

$$I = \frac{m}{\pi(b^2 - a^2)} \int_a^b 2\pi\rho d\rho \cdot \rho^2 = \frac{m}{2}\left(b^2 + a^2\right)$$

And the angular momentum is $\boldsymbol{L} = I\boldsymbol{\omega} = \frac{m}{2}\left(b^2 + a^2\right)\omega \boldsymbol{e}_z$. Thus, the gyromagnetic ratio is

$$\gamma = \frac{\boldsymbol{\mu}}{\boldsymbol{L}} = \frac{Q}{2m}.$$

For the case that the magnetic field is applied along the y-direction, the torque acted on the disk is

$$\boldsymbol{M} = \boldsymbol{\mu} \times \boldsymbol{B}.$$

The equation of the motion is then

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{M} = \boldsymbol{\mu} \times \boldsymbol{B} = \gamma \boldsymbol{L} \times \boldsymbol{B} = -\gamma \boldsymbol{B} \times \boldsymbol{L}$$

Therefore, the angular velocity of precession is

$$\boldsymbol{\Omega} = -\gamma \boldsymbol{B} = -\frac{qB}{2m}\boldsymbol{e}_y.$$

6. (20pts) A circular parallel-plate capacitor of radius R and plate separation d is connected to a source of potential difference $V = V_0 \cos \omega t$. The edge effects can be ignored since $d << R$.

(a) Find the electric field inside the capacitor and the area densities of the electric charge on the plates as a function of time.

(b) Find the magnetic field inside the capacitor as a function of time and the distance to the axis of the plate.

(c) Find the electromagnetic field energy flow density (the Poynting vector) through the lateral surface of the capacitor.

Solution: (a) Ignoring the edge effects, the electric field inside the capacitor is

$$E_z = -V/d = -V_0/d \cos \omega t.$$

And the area densities on the upper and lower plates are

$$\sigma = \pm \varepsilon_0 E = \pm \frac{\varepsilon_0 V_0}{d} \cos \omega t.$$

(b) Making a circular path of radius $\rho \le R$ that is concentric with the circular capacitor, the Ampere-Maxwell equation gives

$$2\pi\rho B_\phi = \mu_0 \varepsilon_0 \pi \rho^2 \frac{\partial E_z}{\partial t} = \frac{\mu_0 \varepsilon_0 \pi \rho^2 V_0 \omega}{d} \sin \omega t \quad \text{or} \quad B_\phi = \frac{\mu_0 \varepsilon_0 \rho V_0 \omega}{2d} \sin \omega t.$$

(c) The Poynting vector at the lateral surface is

$$\boldsymbol{S} = \frac{\left(\boldsymbol{E} \times \boldsymbol{B}\right)_{\rho=R}}{\mu_0} = \frac{\varepsilon_0 V_0^2 \omega R}{4d^2} \sin 2\omega t \boldsymbol{e}_\rho.$$