[流体力学]

一、曲面坐标系

$H_i^2=\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2$

$\nabla u=\left(\frac{1}{H_1}\frac{\partial u}{\partial q_1},\frac{1}{H_2}\frac{\partial u}{\partial q_2},\frac{1}{H_3}\frac{\partial u}{\partial q_3}\right)$

$\nabla\cdot\vec{A}=\frac{1}{H_1H_2H_3}\sum_i\frac{\partial}{\partial q_i}\left(\frac{H_1H_2H_3}{H_i}A_i\right)$

$(\nabla\times\vec{A})_1=\frac{1}{H_2H_3}\left[\frac{\partial(H_3A_3)}{\partial q_2}-\frac{\partial(H_2A_2)}{\partial q_3}\right]$

二、流体描述

1. Lagrange

$\vec{r}=\vec{r}(a,b,c,t)$ 流体质点

Euler

$\vec{v}=\vec{v}(\vec{r},t)$ 场

理想正压

外力有势

$\begin{cases}\nabla\Pi=\frac{\nabla p}{\rho}\\ \vec{F}=-\nabla V\end{cases}$

2. 随体导数

$\frac{d\vec{v}}{dt}=\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}$

$\Rightarrow\frac{d}{dt}=\frac{\partial}{\partial t}+(\vec{v}\cdot\nabla)$

3. 轨迹

$\frac{d\vec{r}}{dt}=\vec{v}\Rightarrow\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{dz}{v_z}=t$

t为自变量

4. 流线

$d\vec{r}\times\vec{v}=0\Rightarrow\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{dz}{v_z}$

微元与速度平行 t为参数

5. 速度分解

(Helmholtz)

$\vec{v}=\vec{v_0}+\frac{1}{2}\underbrace{(\nabla\times\vec{v})}_{\vec{\Omega}}\times\delta\vec{r}+\nabla\psi$

平动 转动 变形

6. 涡旋

$d\vec{r}\times\vec{\Omega}=0$ 涡线

球：$d\vec{r}=dr\vec{e_r}+rd\theta\cdot\vec{e_\theta}+r\sin\theta d\varphi\cdot\vec{e_\varphi}$

$\Rightarrow$ 涡管任一截面涡通量守恒

① (Stokes)

$I\Leftarrow\iint_S\vec{\Omega}\cdot d\vec{S}=\oint_C\vec{v}\cdot d\vec{l}\rightarrow\Gamma$

运动方程得到

②

$\frac{d\vec{\Omega}}{dt}=-(\vec{\Omega}\cdot\nabla)\vec{v}+\vec{\Omega}(\nabla\cdot\vec{v})=0$ (Helmholtz)

$\frac{d\Gamma}{dt}=0$，$\frac{dI}{dt}=0$ (Kelvin)

某时刻无(有)旋，前/后任一时刻无(有)旋 (Lagrange)

$\frac{d}{dt}\left(\frac{\vec{\Omega}}{\rho}\right)\parallel\frac{d}{dt}\delta\vec{r}$ (Helmholtz)

三、基本方程组

1. 连续性方程

① Lagrange观点

$\frac{d\left(\int_\tau\rho\,\delta\tau\right)}{dt}=0$

$\Rightarrow\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$

② Euler观点

$\oiint_S\rho\vec{v}\cdot\delta\vec{S}=-\int_\tau\frac{\partial\rho}{\partial t}\delta\tau$

不可压缩为 $\nabla\cdot\vec{v}=0$

运动方程

(理想)

动量定理

$$dxdz\,p|_y - dxdz\,p|_{y+dy} + \rho f\,dxdydz = \rho\,dxdydz\frac{dV_y}{dt}$$

$$\Rightarrow \rho\frac{d\vec v}{dt} = -\nabla p + \rho\vec f$$

3、能量方程

$$\rho\frac{d}{dt}\left(u+\frac{v^2}{2}\right) = \rho\vec F\cdot\vec v + \nabla\cdot(\vec{\vec p}\cdot\vec v)+\nabla\cdot(k\nabla T)+\rho q$$

内 动 质量力 面力 传导 辐射

4. 本构方程

(广义 Newton)

$$\vec{\vec P} = -p\vec{\vec I} + 2\mu\left[\vec{\vec S} - \frac{1}{3}(\nabla\cdot\vec v)\vec{\vec I}\right]$$

5. 状态方程

$$p = f(T,\rho)$$

⇒理想可压缩

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec v)=0$$

$$\rho\frac{d\vec v}{dt}=\rho\vec F-\nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right)=0$$

四、流体静力学

1. $\Rightarrow \rho\vec F=\nabla p$

$p=f(\rho,T)$

$\rho\frac{du}{dt}=\nabla\cdot(k\nabla T)$

↳ $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right)=0$

且 $\vec F\cdot(\nabla\times\vec F)=0$

⇒(Pascal)

$\nabla(p+\rho\vec V)=0$

$\Rightarrow p-\rho gz=const=p_0$

⇒(Archimedes)

$\vec R=-\vec G$ 大小

$\vec L=\vec r_c\times\vec R$ 作用线

2. Bernoulli 积分

$\frac{d\vec v}{dt}=-\frac{1}{\rho}\nabla p+\vec F$ 理想正压，外力有势

$\downarrow$ $\quad -\nabla\Pi \quad -\nabla\vec V$

$\frac{\partial\vec v}{\partial t}+(\vec v\cdot\nabla)\vec v$

↳ $\nabla\left(\frac{v^2}{2}\right)+(\nabla\times\vec v)\times\vec v$

$$\Rightarrow \frac{\partial\vec v}{\partial t}+\nabla\left(\frac{v^2}{2}+\Pi+\vec V\right)+(\nabla\times\vec v)\times\vec v=0$$

①无旋 $\vec v=\nabla\varphi$

$$\frac{\partial\varphi}{\partial t}+\frac{v^2}{2}+\Pi+\vec V=f(t)$$

②定常 $\frac{v^2}{2}+\Pi+\vec V=C(\psi)$ → 标识流线

③ ①+② $\frac{v^2}{2}+\Pi+\vec V=const$

五、理想不可压缩流体无旋

1.
$$\begin{cases}\nabla\cdot\vec{v}=0\\ \dfrac{d\vec{v}}{dt}=\vec{F}-\dfrac{1}{\rho}\nabla P\end{cases}\xrightarrow{\text{外力有势}}\begin{cases}\nabla\cdot\vec{v}=0\\ \dfrac{\partial\varphi}{\partial t}+\dfrac{v^2}{2}+\dfrac{P}{\rho}+V=f(t)\end{cases}$$

平面运动
$$\begin{cases}\omega_z=0\\ \dfrac{\partial}{\partial z}=0\end{cases}$$

$\boxed{\text{平+无旋}}\rightarrow$ 速度势 φ $\boxed{\text{平+不可压}}\rightarrow$ 流函数 ψ

$\Downarrow$ 平+无旋+不可压
Laplace方程

2. 复位势 $w(z)=\varphi+i\psi$

$$\begin{cases}\dfrac{\partial\varphi}{\partial x}=\dfrac{\partial\psi}{\partial y}\\ \dfrac{\partial\varphi}{\partial y}=-\dfrac{\partial\psi}{\partial x}\end{cases}$$

且 $\nabla\varphi\cdot\nabla\psi=0$

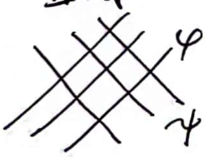
$$\frac{dw}{dz}=u-iv$$

$$\Gamma+iQ=\oint_c\frac{dw}{dz}dz$$

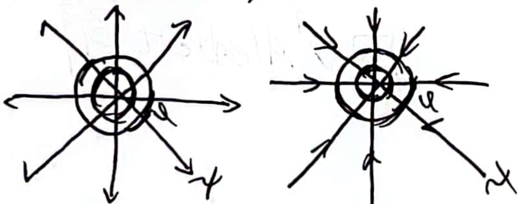
∘ 驻点：$\dfrac{dw}{dz}=0$

3. 奇点法

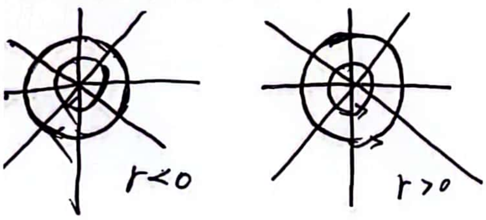
①
$w=az,\ a\in\mathbb{C}$
直线



②
$w=a\ln z,\ a\in\mathbb{R}$



③
$w=ib\ln z,\ b\in\mathbb{R}$



涡

④
$w=\dfrac{c}{z}\quad c\in\mathbb{C}$

偶极子

$M=me^{i\beta}$

$$\begin{cases}\varphi=-\dfrac{M}{2\pi}\dfrac{x}{x^2+y^2}\\ \psi=\dfrac{M}{2\pi}\dfrac{y}{x^2+y^2}\end{cases}$$

eg.
Rankine流 直线+点源

圆柱无环量绕流

$$w(z)=V_\infty z+\frac{V_m}{2\pi}\frac{1}{z}$$

4. Blasius
流体作用在单位柱体合力
$$F_x-iF_y=\frac{1}{2}i\rho\oint_c\left(\frac{dw}{dz}\right)^2dz$$

$\nabla^2\phi=0$ 球坐标（与φ无关）

$$\phi\sim\sum_l\left(A_lr^l+B_lr^{-l-1}\right)P_l(\cos\theta)$$

$$\phi\sim\sum_{l=0}^{m}\sum_{m=-l}^{l}\left(A_{lm}r^l+B_{lm}r^{-l-1}\right)e^{im\varphi}P_l^m(\cos\theta)$$

5. D'Alembert佯谬
没考虑粘性对柱产生摩擦阻力及由于边界层分离所产生压差阻力

6. 有环量绕流
$$w(z)=V_\infty\left(z+\frac{a^2}{z}\right)-\frac{\Gamma}{2\pi i}\ln z$$

$$R_y=\rho V_\infty\Gamma$$

(Жуковский)

六、理想不可压缩流体波动理论

$z=\zeta(x,y,t)$

1、方程　　$a \ll \lambda$
振幅

$$\Rightarrow \begin{cases} \Delta\varphi = 0 \\ \dfrac{p-p_0}{\rho} = -\dfrac{\partial\varphi}{\partial t} - gz \end{cases}$$

2、

平面波周期解

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

$$\begin{cases} z=0 \quad \dfrac{\partial\varphi}{\partial z} = -\dfrac{1}{g}\dfrac{\partial^2\varphi}{\partial t^2} \\ z=-\infty, \quad \dfrac{\partial\varphi}{\partial z} = 0 \\ t=0, z=0, \quad \varphi = F(x), \dfrac{\partial\varphi}{\partial t} = f(x) \end{cases}$$

$$\frac{T''}{T} = -\sigma^2 \qquad \frac{X''}{X} = -\frac{Z''}{Z} = -k^2$$

$$T \sim \begin{Bmatrix} \cos\sigma t \\ \sin\sigma t \end{Bmatrix} \qquad X \sim \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix}$$

$$Z \sim \begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix} \Rightarrow gk = \sigma^2$$

色散关系

$$\begin{cases} \lambda = \dfrac{2\pi}{k} \\ T = \dfrac{2\pi}{\sigma} \end{cases}$$

$c \sim \dfrac{\sigma}{k}$ 相速

$\dfrac{d\sigma}{dk}$ 群速

3. 界面扰动

$z=0$ 　ρ_1

ρ_2

进波解

$$\varphi_1' = C_1 e^{-kz} e^{i(kx-\sigma t)}$$

$$\varphi_2' = C_2 e^{kz} e^{i(kx-\sigma t)}$$

分界面

$$z = Ae^{i(kx-\sigma t)}$$

用 $\dfrac{\partial\varphi'}{\partial z} = \dfrac{d\zeta}{dt} = \dfrac{\partial\zeta}{\partial t} + \dfrac{dx}{dt}\dfrac{\partial\zeta}{\partial x}$

及 P_1 与 P_2 关系

(如 $P_1 = P_2 = P_0'$) 定系数

4. 长波

$\lambda \gg h \gg a$

$$\frac{\partial^2 u}{\partial t^2} - gh\frac{\partial^2 u}{\partial x^2} = 0, \quad c = \sqrt{gh}$$

用 D'Alembert 解